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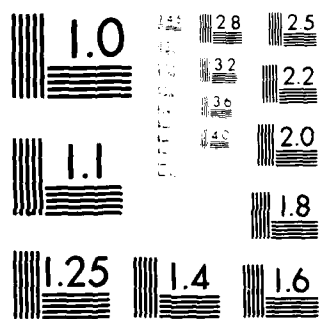
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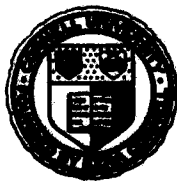
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TECHNICAL REPORT NO. 532

April 1982

ON THE PERFORMANCE CHARACTERISTICS OF
A CLOSED ADAPTIVE SEQUENTIAL PROCEDURE
FOR SELECTING THE BEST BERNOULLI POPULATION

by

Robert E. Bechhofer
and
Radhika V. Kulkarni*

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*Curriculum in Operations Research and Systems Analysis, University of North Carolina, Chapel Hill.

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Abstract

In a recent paper, Bechhofer and Kulkarni proposed closed adaptive sequential procedures for a general class of k -population Bernoulli selection goals. These sequential selection procedures achieve the same probability of a correct selection, uniformly in the unknown single-trial "success" probabilities p_i ($1 \leq i \leq k$), as do the corresponding single-stage selection procedures which take exactly n observations from each of the k populations. The sequential procedures always require less (often substantially less) than kn observations to terminate experimentation. This earlier paper described the procedures, discussed their performance in general terms, and cited several of their optimality properties.

In the present paper we specialize these procedures, and focus on the particular goal of selecting the population associated with $p_{[k]}$ where $p_{[1]} \leq \dots \leq p_{[k]}$ are the ordered p_i ($1 \leq i \leq k$). We give exact numerical results for such performance characteristics of the sequential procedure (P^*) as the distribution of the total number of observations $N_{(i)}$ taken from the population associated with $p_{[i]}$ ($1 \leq i \leq k$), and the total number of observations $N = \sum_{i=1}^k N_{(i)}$ taken from all k populations, when the procedure terminates sampling. A simple upper bound for $E\{N_{(i)}\}$ ($i \neq k$) is given. These results along with other related ones will assist the potential user of the sequential procedure in assessing its merits relative to those of other competing procedures.

Key words:

Bernoulli selection problem, clinical trials, selection procedures, ranking procedures, sequential analysis, adaptive procedures, stationary sampling procedures, least-failures sampling procedures, two-population optimal sampling procedure.

1. Introduction and summary

Let π_i ($1 \leq i \leq k$) denote $k \geq 2$ Bernoulli populations with corresponding single-trial "success" probabilities p_i . Denote the ordered values of the p_i by $p_{[1]} \leq \dots \leq p_{[k]}$; the values of the p_i and the pairing of the π_i with the $p_{[j]}$ ($1 \leq i, j \leq k$) are assumed to be completely unknown. The goal of the experimenter is to select the population associated with $p_{[k]}$.

Sobel and Huyett [1957] proposed a single-stage selection procedure for this problem; their procedure involves taking exactly n observations from each of the k populations. In a recent paper, Bechhofer and Kulkarni [1982] (hereinafter referred to as B-K), the authors proposed a sequential selection procedure employing a closed one-at-a-time adaptive sampling rule for this same problem; their procedure, which takes no more than n observations from any one of the k populations, achieves the same probability of a correct selection as does the single-stage procedure uniformly in $p = (p_1, \dots, p_k)$. The sequential procedure was shown in B-K to have certain optimal properties for $k = 2$, and in addition to have various desirable properties for general selection goals when $k \geq 2$.

In the present paper we focus on the particular goal of selecting the population associated with $p_{[k]}$. For that goal we give formulae and exact numerical results for such performance characteristics of the sequential procedure as the distribution of the total number of observations $N_{(i)}$ taken from the population associated with $p_{[i]}$ ($1 \leq i \leq k$), and the total number of observations $N = \sum_{i=1}^k N_{(i)}$ taken from all k populations, when the procedure terminates sampling. These fundamental performance characteristics, along with the exact achieved probability of a correct selection ($P\{CS\}$), can assist the potential user of the procedure in assessing its merits relative to those of other competing procedures.

The plan of the paper is as follows: In Section 2 we describe our sequential selection procedure P^* . A number of examples are given to illustrate how it operates. In Section 3 we cite several of the optimal properties of P^* for $k = 2$, and point out general desirable properties of P^* for $k \geq 2$. In addition we conjecture certain optimal properties of P^* for $k > 2$. All of these properties were considered at length in B-K and in Kulkarni [1981]. They are set forth here in order to serve as reference points when we discuss in Section 5 the calculated performance characteristics of P^* .

Our results concerning such performance characteristics are given in considerable detail in Section 4. These results constitute the heart of the present paper. We show in Section 4 how to calculate the exact $P\{CS\}$ and the exact distribution of $N_{(i)}$ ($1 \leq i \leq k$) and N for P^* ; the computational complexity of such calculations as k and/or n increases is stressed. A comprehensive set of tables containing our calculated results is provided. These give an excellent over-all picture of the performance of P^* . We study and discuss the performance characteristics in Section 5. Derivations of general formulae used for calculating some of the $P\{CS\}$, $E\{N\}$, and $E\{N_{(i)}\}$ results for P^* are given in the Appendices.

2. The sequential selection procedure (P^*)

We shall use the following terminology and notation: By stage m we mean that a total of m observations have been taken. Let S_i^m (F_i^m) denote a success (failure) from Π_i at stage m ($1 \leq i \leq k$, $1 \leq m \leq kn-1$). Let $n_{i,m}$ denote the total number of observations taken from Π_i through stage m , and let $z_{i,m}$ denote the total number of successes yielded by Π_i through stage m ($1 \leq i \leq k$, $1 \leq m \leq kn-1$).

Our sequential procedure $P^* = (R^*, S^*, T^*)$, described below, takes no more than n observations from any of the k populations. The basis for

specifying n (e.g., to guarantee an indifference-zone probability requirement as in Sobel and Huyett [1957], equations (5) or (13), or because of availability of observations or because of other economic considerations) is of no concern to us here.

PROCEDURE (P)* FOR SELECTING THE POPULATION ASSOCIATED WITH $P_{[k]}$:

Sampling rule (R*): At stage m ($0 \leq m \leq kn-1$), take the next observation from the population which has the smallest number of failures among all π_i for which $n_{i,m} < n$ ($1 \leq i \leq k$). If there is a tie among such equal-number-of-failure populations, take the next observation from that one of them which has the largest number of successes. If there is a further tie among such equal-number-of-success populations, select one of them at random and take the next observation from it. (2.1)

Stopping rule (S*): Stop sampling at the first stage m at which there exists at least one population π_i satisfying (2.2)

$$z_{i,m} \geq z_{j,m} + n - n_{j,m} \text{ for all } j \neq i \text{ } (1 \leq i, j \leq k).$$

Terminal decision rule (T*): If $r \geq 1$ populations, say $\pi_{i_1}, \dots, \pi_{i_r}$ simultaneously satisfy (2.2), then select one of them at random as associated with $P_{[k]}$. (2.3)

Remark 2.1: The l.h.s. of (2.2) represents the current total number of successes from π_i while the r.h.s. represents the current total number of successes from π_j plus the total number of potential successes from π_j if all of the remaining observations $(n - n_{j,m})$ from π_j after stage m are successes. Hence, (2.2) tells us to stop sampling as soon as there exists one or more populations which have at least as many successes as the maximum possible number of successes at termination from any of the other populations.

To illustrate the sequential procedure p^* , we give the following stopping sequences:

Example 2.1: For $(k = 3, n = 2)$, stop if

$$\begin{array}{ccc} \underline{\pi_1} & \underline{\pi_2} & \underline{\pi_3} \\ & & s_3^1 \\ & & s_3^2 \end{array} .$$

Then π_3 satisfies (2.2). Hence, select π_3 as associated with $p_{[3]}$.

Example 2.2: For $(k = 3, n = 2)$, stop if

$$\begin{array}{ccc} \underline{\pi_1} & \underline{\pi_2} & \underline{\pi_3} \\ f_1^3 & f_2^2 & f_3^1 \\ f_1^4 & & f_3^5 \end{array} .$$

Then π_2 satisfies (2.2). Hence, select π_2 as associated with $p_{[3]}$.

Example 2.3: For $(k = 3, n = 2)$, stop if

$$\begin{array}{ccc} \underline{\pi_1} & \underline{\pi_2} & \underline{\pi_3} \\ s_1^3 & s_2^1 & s_3^5 \\ f_1^4 & f_2^2 & \end{array} .$$

Then π_3 satisfies (2.2). Hence, select π_3 as associated with $p_{[3]}$.

Example 2.4: For $(k = 3, n = 2)$, stop if

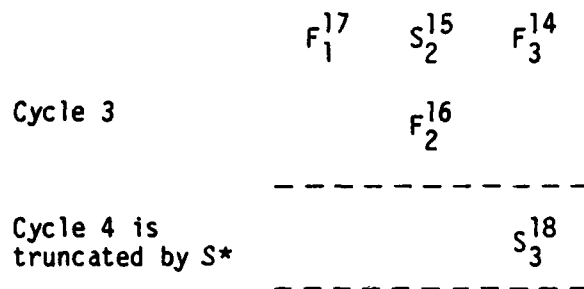
π_1	π_2	π_3
S_1^3	S_2^1	F_3^5
F_1^4	F_2^2	

Then π_1 and π_2 satisfy (2.2). Hence, select one of them at random as associated with $P_{[3]}$.

Remark 2.2: We can regard the sampling for P^* as proceeding in cycles; within each cycle (except perhaps the last one) the outcomes from each population consist of a sequence of successes followed by a single failure. In Example 2.5, below, the fourth cycle is truncated by the stopping rule S^* .

Example 2.5: For $(k = 3, n = 8)$, stop if

	π_1	π_2	π_3
	F_1^3	S_2^1	S_3^4
Cycle 1		F_2^2	S_3^5
			F_3^6
<hr/>			
	S_1^{12}	S_2^{10}	S_3^7
Cycle 2	F_1^{13}	F_2^{11}	S_3^8
			F_3^9
<hr/>			



Then π_3 satisfies (2.2). Hence, select π_3 as associated with $p_{[3]}$.

Remark 2.3: R^* is not a play-the-winner (PW) sampling rule as can be seen from S_3^7 of Example 2.5. R^* is PW within a cycle but may not be PW as sampling proceeds from one cycle to the next. Most of the procedures which have been proposed for the Bernoulli selection problem employ PW sampling rules. The reader is referred to B-K for an extensive bibliography of Bernoulli selection procedures.

Remark 2.4: R^* has been proposed by Kelly [1981] for a Bernoulli multi-armed bandit problem when the discount factor is near one; he refers to it as "the least failures rule."

3. The Performance of P^*

In order to make the present paper self-contained, we cite in Section 3.1 several of the optimal properties of P^* for $k = 2$. In Section 3.2 we mention several desirable properties for $k \geq 2$, and in Section 3.3 we conjecture certain optimal properties for $k > 2$. All of these were considered at length in B-K and in Kulkarni [1981]; the optimal properties for $k = 2$ are proved in Kulkarni [1982].

The fundamental general theorem concerning the $P\{CS\}$ achieved by P^* , cited immediately below, was proved (in greater generality) in B-K. For this

theorem it is assumed that if two or more populations have a common p-value equal to $\max\{p_1, \dots, p_k\}$, then those tied populations are tagged in such a way that their ordering is unique, i.e., one is associated with $p_{[k]}$, a second with $p_{[k-1]}$, etc. In the following theorem we denote by R_{SS} and T_{SS} the sampling rule and the terminal decision rule, respectively, used by the Sobel-Huyett single-stage procedure.

Theorem 3.1:

$$P\{CS|(R_{SS}, T_{SS})\} \equiv P\{CS|(R^*, S^*, T^*)\}$$

uniformly in $p = (p_1, \dots, p_k)$ for all $k \geq 2$.

In Sections 3.1 and 3.3, below, R refers to an arbitrary sampling rule which takes no more than n observations from any of the $k \geq 2$ populations, and which is used in conjunction with S^* and T^* of (2.2) and (2.3), respectively. (Note: R^* is a special case of R ; Theorem 3.1 was actually proved for R in B-K.)

3.1 Optimal properties of R^* for $k = 2$

For $k = 2$ let \bar{R}^* denote the conjugate sampling rule in which $n_{i,m} - z_{i,m}$ and $z_{i,m}$ of (2.1) are replaced by $z_{i,m}$ and $n_{i,m} - z_{i,m}$, respectively, for $i = 1, 2$.

Theorem 3.2: Procedure $P^* = (R^*, S^*, T^*)$ minimizes $E\{N|(p_1, p_2)\}$ for $p_1 + p_2 > 1$ among all procedures (R, S^*, T^*) . Procedure $\bar{P}^* = (\bar{R}^*, S^*, T^*)$ minimizes $E\{N|(p_1, p_2)$ for $p_1 + p_2 < 1$ among all procedures (R, S^*, T^*) . Both P^* and \bar{P}^* minimize $E\{N|(p_1, p_2)\}$ among all procedures (R, S^*, T^*) for $p_1 + p_2 = 1$.

Theorem 3.3: P^* minimizes $E\{N_{(1)} | (p_1, p_2)\}$ for $p_{[2]} \geq 1/2$ among all procedures (R, S^*, T^*) .

Let N^F denote the random total number of "failures" that have occurred from both populations when sampling stops.

Theorem 3.4: P^* minimizes $E\{N^F | (p_1, p_2)\}$ for $p_1 + p_2 \geq 1$, and \bar{P}^* minimizes $E\{N^F | (p_1, p_2)\}$ for $p_{[2]} \leq 1/2$ among all procedures (R, S^*, T^*) .

Remark 3.1: The sampling rules R^* and \bar{R}^* , which for $k = 2$ possess the optimality properties given by Theorems 3.2, 3.3 and 3.4, are stationary, i.e., they are independent of n .

3.2 General properties of P^* for $k > 2$

The following additional desirable properties of P^* are discussed in B-K.

- a) $n \leq N \leq kn-1$, i.e., N is bounded.
- b) $P\{N = n | p\} \rightarrow 1$ for $p_{[1]} \rightarrow 1$, and $P\{N = kn-1 | p\} \rightarrow 1$ for $p_{[k]} \rightarrow 0$.
- c) Populations with small p -values tend to be sampled less frequently than those with large p -values.
- d) No special tables of constants are necessary to carry out P^* for $k \geq 2$, and it is very easy to implement.

3.3 Conjectured optimal property of P^* for $k > 2$

The following conjecture was made (and some supporting results given) in B-K:

Conjecture 3.1: P^* minimizes $E\{N | (p_1, p_2, \dots, p_k)\}$ for $p_{[1]} + p_{[2]} > 1$ among all procedures (R, S^*, T^*) for $k > 2$.

4. Exact performance characteristics of P^*

The properties of P^* mentioned in Section 3 are of a general nature. However, before an experimenter would choose to use P^* he presumably would wish to have detailed quantitative knowledge concerning its performance characteristics in order to compare them with the corresponding properties of competing procedures. In this section we provide such data.

4.1 Formulae

In this section we show for $k = 2$ and 3 and very small n the steps involved in computing the exact $P\{CS\}$ achieved by P^* , and the exact joint distributions of the $N_{(i)}$ ($1 \leq i \leq k$) and the distribution of N . As will quickly become evident, these calculations are extremely tedious, and rapidly get out of hand even for k - and/or n -values as small as four. Thus we were forced to adopt a different approach which we discuss in Section 4.2.

In Tables 4.1, 4.2 and 4.3 we have enumerated all of the termination sequences and associated quantities for P^* for $(k = 2, n = 1)$, $(k = 2, n = 2)$ and $(k = 3, n = 1)$, respectively. Such tables contain the information necessary to derive the formulae for the $P\{CS\}$, the joint distribution of the $N_{(i)}$ ($1 \leq i \leq k$), and the distribution of N . From these we can compute the marginal distribution of the $N_{(i)}$, and $E\{N_{(i)}\}$ ($1 \leq i \leq k$) as well as $E\{N\}$.

Thus, for example, using the information in Table 4.2 for $k = 2, n = 2$ it is straightforward to derive the following exact formulae:

Table 4.1
Termination Sequences and Associated Quantities for P^*
when $k = 2$ and $n = 1$
($p_1 < p_2$)

Termination Sequence	Selection	Is the Selection Correct?	Probability of Termination Sequence	Number of Observations from		
				π_1	π_2	All π
S_1^1	π_1	No	$(1/2)p_1$	1	0	1
F_1^1	π_2	Yes	$(1/2)(1-p_1)$	1	0	1
S_2^1	π_2	Yes	$(1/2)p_2$	0	1	1
F_2^1	π_1	No	$(1/2)(1-p_2)$	0	1	1

$$P\{CS\} = (1/2)(1-p_1) + (1/2)p_2 = (1/2) + (1/2)(p_2-p_1)$$

Table 4.2

Termination Sequences and Associated Quantities for p^*

when $k = 2$ and $n = 2$

$(p_1 < p_2)$

Termination Sequence	Selection	Is the Selection Correct?	Probability of Termination Sequence	Number of Observations from		
				π_1	π_2	All π
$S_1^1 S_1^2$	π_1	No	$(1/2)p_1^2$	2	0	2
$F_1^1 S_2^2$	π_2	Yes	$(1/2)(1-p_1)p_2$	1	1	2
$S_2^1 S_2^2$	π_2	Yes	$(1/2)p_2^2$	0	2	2
$F_2^1 S_1^2$	π_1	No	$(1/2)(1-p_2)p_1$	1	1	2
$S_1^1 F_1^2 S_2^3$	π_2	Yes	$(1/2)p_1(1-p_1)p_2$	2	1	3
$S_1^1 F_1^2 F_2^3$	π_1	No	$(1/2)p_1(1-p_1)(1-p_2)$	2	1	3
$F_1^1 F_2^2 S_1^3$	π_1	No	$(1/2)(1-p_1)(1-p_2)(1/2)p_1$	2	1	3
$F_1^1 F_2^2 F_1^3$	π_2	Yes	$(1/2)(1-p_1)(1-p_2)(1/2)(1-p_1)$	2	1	3
$F_1^1 F_2^2 S_2^3$	π_2	Yes	$(1/2)(1-p_1)(1-p_2)(1/2)p_2$	1	2	3
$F_1^1 F_2^2 F_2^3$	π_1	No	$(1/2)(1-p_1)(1-p_2)(1/2)(1-p_2)$	1	2	3
$S_2^1 F_2^2 S_1^3$	π_1	No	$(1/2)p_2(1-p_2)p_1$	1	2	3
$S_2^1 F_2^2 F_1^3$	π_2	Yes	$(1/2)p_2(1-p_2)(1-p_1)$	1	2	3
$F_2^1 F_1^2 S_1^3$	π_1	No	$(1/2)(1-p_2)(1-p_1)(1/2)p_1$	2	1	3
$F_2^1 F_1^2 F_1^3$	π_2	Yes	$(1/2)(1-p_2)(1-p_1)(1/2)(1-p_1)$	2	1	3
$F_2^1 F_1^2 S_2^3$	π_2	Yes	$(1/2)(1-p_2)(1-p_1)(1/2)p_2$	1	2	3
$F_2^1 F_1^2 F_2^3$	π_1	No	$(1/2)(1-p_2)(1-p_1)(1/2)(1-p_2)$	1	2	3

Table 4.3

Termination Sequences and Associated Quantities for p^*

when $k = 3$ and $n = 1$

$$(p_1 \leq p_2 < p_3)$$

Termination Sequence	Selection	Is the Selection Correct?	Probability of Termination Sequence	Number of Observations from			
				π_1	π_2	π_3	All π
S_1^1	π_1	No	$(1/3)p_1$	1	0	0	1
S_2^1	π_2	No	$(1/3)p_2$	0	1	0	1
S_3^1	π_3	Yes	$(1/3)p_3$	0	0	1	1
$F_1^1 S_2^2$	π_2	No	$(1/3)(1-p_1)(1/2)p_2$	1	1	0	2
$F_1^1 F_2^2$	π_3	Yes	$(1/3)(1-p_1)(1/2)(1-p_2)$	1	1	0	2
$F_1^1 S_3^2$	π_3	Yes	$(1/3)(1-p_1)(1/2)p_3$	1	0	1	2
$F_1^1 F_3^2$	π_2	No	$(1/3)(1-p_1)(1/2)(1-p_3)$	1	0	1	2
$F_2^1 S_1^2$	π_1	No	$(1/3)(1-p_2)(1/2)p_1$	1	1	0	2
$F_2^1 F_1^2$	π_3	Yes	$(1/3)(1-p_2)(1/2)(1-p_1)$	1	1	0	2
$F_2^1 S_3^2$	π_3	Yes	$(1/3)(1-p_2)(1/2)p_3$	0	1	1	2
$F_2^1 F_3^2$	π_1	No	$(1/3)(1-p_2)(1/2)(1-p_3)$	0	1	1	2
$F_3^1 S_1^2$	π_1	No	$(1/3)(1-p_3)(1/2)p_1$	1	0	1	2
$F_3^1 F_1^2$	π_2	No	$(1/3)(1-p_3)(1/2)(1-p_1)$	1	0	1	2
$F_3^1 S_2^2$	π_2	No	$(1/3)(1-p_3)(1/2)p_2$	0	1	1	2
$F_3^1 F_2^2$	π_1	No	$(1/3)(1-p_3)(1/2)(1-p_2)$	0	1	1	2

Probability of a correct selection

$$P\{CS\} = 1/2 + (p_2 - p_1)[1 - (1/2)(p_1 + p_2) + p_1 p_2]$$

Distribution of $N_{(1)}$

$$P\{N_{(1)} = 0\} = (1/2)p_2^2$$

$$P\{N_{(1)} = 1\} = (1/2)(1 - p_2^2 - p_1 p_2 + p_2)$$

$$P\{N_{(1)} = 2\} = (1/2)(1 + p_1 p_2 - p_2)$$

$$E\{N_{(1)}\} = (1/2)(3 - p_2^2 + p_1 p_2 - p_2)$$

Distribution of $N_{(2)}$

$$P\{N_{(2)} = 0\} = (1/2)p_1^2$$

$$P\{N_{(2)} = 1\} = (1/2)(1 - p_1^2 - p_1 p_2 + p_1)$$

$$P\{N_{(2)} = 2\} = (1/2)(1 + p_1 p_2 - p_1)$$

$$E\{N_{(2)}\} = (1/2)(3 - p_1^2 + p_1 p_2 - p_1)$$

Distribution of N

$$P\{N = 2\} = (1/2)[p_1 + p_2 + (p_1 - p_2)^2]$$

$$P\{N = 3\} = (1/2)[2 - p_1 - p_2 - (p_1 - p_2)^2]$$

$$E\{N\} = (1/2)[6 - p_1 - p_2 - (p_1 - p_2)^2]$$

$$E\{N | p_1 = p_2 = p\} = 3 - p$$

Similarly, from Table 4.3 for $k = 3, n = 1$ we obtain

Probability of a correct selection

$$P\{CS\} = 1/3 + (1/3)(2p_3 + p_1 p_2 - p_1 - p_2) - (1/6)p_3(p_1 + p_2)$$

Distribution of $N_{(1)}$

$$P\{N_{(1)} = 0\} = 1/3 + (1/6)(p_2 + p_3)$$

$$P\{N_{(1)} = 1\} = 2/3 - (1/6)(p_2 + p_3)$$

$$E\{N_{(1)}\} = 2/3 - (1/6)(p_2 + p_3)$$

Distribution of $N_{(2)}$

$$P\{N_{(2)} = 0\} = 1/3 + (1/6)(p_1+p_3)$$

$$P\{N_{(2)} = 1\} = 2/3 - (1/6)(p_1+p_3)$$

$$E\{N_{(2)}\} = 2/3 - (1/6)(p_1+p_3)$$

Distribution of $N_{(3)}$

$$P\{N_{(3)} = 0\} = 1/3 + (1/6)(p_1+p_2)$$

$$P\{N_{(3)} = 1\} = 2/3 - (1/6)(p_1+p_2)$$

$$E\{N_{(3)}\} = 2/3 - (1/6)(p_1+p_2)$$

Distribution of N

$$P\{N = 1\} = (1/3)(p_1+p_2+p_3)$$

$$P\{N = 2\} = 1 - (1/3)(p_1+p_2+p_3)$$

$$E\{N\} = 2 - (1/3)(p_1+p_2+p_3)$$

$$E\{N|p_1 = p_2 = p_3 = p\} = 2-p$$

As can be seen, such exact calculations are extremely tedious and rapidly get out of hand as n and/or k increases. (The formula for the $P\{CS\}$ and for $E\{N\}$ is a polynomial of degree $kn-1$ in p_1, p_2, \dots, p_k when the p_i ($1 \leq i \leq k$) are all unequal.) Thus we have derived the formulae for the $P\{CS\}$, $E\{N_{(i)}\}$ ($1 \leq i \leq k$), and $E\{N\}$ for P^* , only for $n \leq 3$. We give here our results for $E\{N\}$ when $p_1 = p_2 = \dots = p_k = p$ (say).

$$(k = 2, n = 1): 1$$

$$(k = 2, n = 2): 3-p$$

$$(k = 2, n = 3): 5-p-3p^2+4p^3-2p^4$$

$$(k = 3, n = 1): 2-p$$

$$(k = 3, n = 2): 5-5p+6p^2-6p^3+2p^4$$

$$(k = 3, n = 3): 8-8p+17p^2-51p^3+93p^4-91p^5+41p^6-6p^7$$

$$(k = 4, n = 1): 3-3p+p^2$$

4.2 Exact calculations

In order to study the performance of p^* for $k = 2$ and 3 we have calculated the $P\{CS\}$, the distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($1 \leq i \leq k$), and the distribution of $N = \sum_{i=1}^k N_{(i)}$, and $E\{N\}$ for p^* for given (k,n) and selected values of $p = (p_1, \dots, p_k)$.

In Section 4.1 we showed how one can enumerate all of the termination sequences for p^* for given (k,n) ; from these one can calculate the exact joint probability distribution of the $N_{(i)}$ ($1 \leq i \leq k$) and the exact probability distribution of N for selected values of p . However, the number of termination sequences increases so rapidly with k and n that very quickly one can exceed the storage and time limitations of even a high-speed computer used to enumerate such sequences. For example, the number of termination sequences for $(k = 2, n = 1)$, $(k = 2, n = 2), \dots, (k = 2, n = 6)$ is 4, 16, 64, 260, 1068, 4420, respectively. (See Tables 4.1 and 4.2 for $(k = 2, n = 1)$ and $(k = 2, n = 2)$, respectively.) Thus we were led to employ other methods.

To calculate the $P\{CS\}$, recall from Theorem 3.1 that $P\{CS|p^*\} = P\{CS|(R_{SS}, T_{SS})\}$ uniformly in p . Hence we can use the computer with the exact formulae for $P\{CS|(R_{SS}, T_{SS})\}$ to evaluate $P\{CS|p^*\}$. The formula for the $P\{CS\}$ is particularly simple when $p_{[1]} = \dots = p_{[k-1]} = p_{[k]}^{-\Delta}$, and is given in Appendix A; calculations based on this formula are contained in Tables 4.11B and 4.21B for $k = 2$ and $k = 3$, respectively. We do not exhibit in this paper the formula for the $P\{CS\}$ for arbitrary p -values ($p_{[k-1]} < p_{[k]}$); calculations based on it are contained in Table 4.24B.

In Appendix C we derive recursion formulae for computing $E\{N\}$ for $k = 2$. (Such formulae can be derived analogously for $k > 2$.) Although we

were not able to solve these recursion equations to obtain a closed form expression for $E\{N\}$, we did employ them to determine $E\{N\}$ recursively using a computer. The method proved to be fairly efficient, and we obtained $E\{N\}$ results for $k = 2, n \leq 100$ and $k = 3, n \leq 40$. These are contained in Tables 4.7 and 4.11A for $k = 2$, and in Tables 4.18, 4.21A and 4.24A for $k = 3$.

Similar recursion formulae were used to calculate $E\{N_{(i)}\}$ ($i = 1, 2, \dots, k$) for $k = 2, n \leq 100$ and $k = 3, n \leq 40$. These results are contained in Table 4.15 for $k = 2$ and in Table 4.27 for $k = 3$. The probability distributions of N and $N_{(i)}$ ($i = 1, 2, \dots, k$) were computed in like manner. For $k = 2$, results are given for the distribution of N for $n = 5$ in Tables 4.4 and 4.8, for $n = 10$ in Tables 4.5 and 4.9, and for $n = 20$ in Tables 4.6 and 4.10; results for the distribution of $N_{(i)}$ ($i = 1, 2$) for $n = 5$ and 10 are given in Tables 4.12 and 4.13, respectively. For $k = 3$, results are given for the distribution of N for $n = 5$ in Tables 4.16, 4.19, 4.22, and for $n = 7$ in Tables 4.17, 4.20, 4.23; results for the distribution of $N_{(i)}$ ($1 \leq i \leq 3$) for $n = 5$ and 7 are given in Tables 4.25 and 4.26, respectively. For $k = 3$ we were not able to obtain the distribution of N or the distribution of $N_{(i)}$ ($1 \leq i \leq 3$) for $n > 7$ since the computing costs would have been prohibitive.

4.2.1 Results for $k = 2$

Tables 4.4, 4.5 and 4.6 give the exact distribution of N , and $E\{N\}$ for $k = 2$ and $n = 5, 10$ and 20 , respectively, when $p_{[1]} = p_{[2]} = 0.1(0.2)0.9$; Table 4.7 gives $E\{N\}$ for $k = 2$ and $n = 5(5)100$ when $p_{[1]} = p_{[2]} = 0.1(0.2)0.9$. Analogously, Tables 4.8, 4.9 and 4.10 give the exact distribution of N , and $E\{N\}$ for $k = 2$ and $n = 5, 10$ and 20 , respectively, when $p_{[1]}+0.2 = p_{[2]} = 0.2, 0.3(0.2)0.9$; Tables 4.11A and 4.11B give $E\{N\}$ and the $P\{CS\}$, respectively, for $k = 2$ and $n = 5(5)100$ when $p_{[1]}+0.2 = p_{[2]} = 0.2, 0.3(0.2)0.9$.

Tables 4.12, 4.13 and 4.14 give the exact distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2$) for $k = 2$ and $n = 5, 10$ and 20 , respectively, when $p_{[1]}^{+0.2} = p_{[2]} = 0.2, 0.3(0.2)0.9, 1.0$. Table 4.15 gives $E\{N_{(i)}\}$ ($i = 1, 2$) for $k = 2$ and $n = 5(5)100$ when $p_{[1]}^{+0.2} = p_{[2]} = 0.2, 0.3(0.2)0.9, 1.0$.

4.2.2 Results for $k = 3$

Tables 4.16 and 4.17 give the exact distribution of N , and $E\{N\}$ for $k = 3$ and $n = 5$ and 7 , respectively, when $p_{[1]} = p_{[2]} = p_{[3]} = 0.1(0.2)0.9$; Table 4.18 gives $E\{N\}$ for $k = 3$ and $n = 2(2)40$ when $p_{[1]} = p_{[2]} = p_{[3]} = 0.1(0.2)0.9$. Analogously, Tables 4.19 and 4.20 give the exact distribution of N , and $E\{N\}$ for $k = 3$ and $n = 5$ and 7 , respectively, when $p_{[1]}^{+0.2} = p_{[2]}^{+0.2} = p_{[3]} = 0.2, 0.3(0.2)0.9$; Table 4.21A gives $E\{N\}$ for $k = 3$ and $n = 2(2)40$ while Table 4.21B gives the $P\{CS\}$ for $k = 3$ and $n = 2(2)50$, both tables being for $p_{[1]}^{+0.2} = p_{[2]}^{+0.2} = p_{[3]} = 0.2, 0.3(0.2)0.9$. In addition, Tables 4.22 and 4.23 give the exact distribution of N , and $E\{N\}$ for $k = 3$ and $n = 5$ and 7 , respectively, when $p_{[1]}^{+\Delta} = p_{[2]} = 0.6 = p_{[3]}^{-\Delta}$ for $\Delta = 0.0(0.1)0.4$; Tables 4.24A and 4.24B give $E\{N\}$ and the $P\{CS\}$, respectively, for $k = 3$ and $n = 2(2)40$ and $n = 2(2)50$, respectively, when $p_{[1]}^{+\Delta} = p_{[2]} = 0.6 = p_{[3]}^{-\Delta}$ for $\Delta = 0.0(0.1)0.4$.

Tables 4.25 and 4.26 give the exact distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($1 \leq i \leq 3$) for $k = 3$ and $n = 5$ and 7 , respectively, when $p_{[1]}^{+\Delta} = p_{[2]} = 0.6 = p_{[3]}^{-\Delta}$ for $\Delta = 0.1(0.1)0.4$. Table 4.27 gives $E\{N_{(i)}\}$ ($1 \leq i \leq 3$) for $k = 3$ and $n = 2(2)40$ when $p_{[1]}^{+\Delta} = p_{[2]} = 0.6 = p_{[3]}^{-\Delta}$ for $\Delta = 0.1(0.1)0.4$.

Table 4.4

Distribution of N , and $E\{N\}$ for p^* when

$k = 2, n = 5$ and $p_{[2]} = p_{[1]}$

a	$P\{N = a\}$ for				
	$p_{[2]} = 0.1$	$p_{[2]} = 0.3$	$p_{[2]} = 0.5$	$p_{[2]} = 0.7$	$p_{[2]} = 0.9$
5	0.000	0.008	0.063	0.240	0.656
6	0.001	0.043	0.156	0.233	0.112
7	0.018	0.151	0.234	0.203	0.093
8	0.151	0.320	0.273	0.176	0.076
9	0.830	0.479	0.273	0.148	0.063
$E\{N\}$	8.809	8.219	7.539	6.759	5.777

Table 4.5

Distribution of N , and $E\{N\}$ for P^* when

$k = 2$, $n = 10$ and $p_{[2]} = p_{[1]}$

a	$P\{N = a\}$ for				
	$p_{[2]} = 0.1$	$p_{[2]} = 0.3$	$p_{[2]} = 0.5$	$p_{[2]} = 0.7$	$p_{[2]} = 0.9$
10	0.000	0.000	0.002	0.040	0.387
11	0.000	0.000	0.010	0.075	0.105
12	0.000	0.001	0.027	0.092	0.093
13	0.000	0.007	0.054	0.106	0.082
14	0.000	0.024	0.087	0.116	0.073
15	0.002	0.060	0.122	0.121	0.065
16	0.013	0.118	0.153	0.122	0.058
17	0.065	0.191	0.175	0.118	0.052
18	0.233	0.264	0.185	0.110	0.046
19	0.687	0.334	0.185	0.098	0.040
$E\{N\}$	18.589	17.585	16.476	14.972	12.586

Table 4.6

Distribution of N , and $E\{N\}$ for P^* when

$k = 2, n = 20$ and $p_{[2]} = p_{[1]}$

a	$P\{N = a\}$ for				
	$p_{[2]} = 0.1$	$p_{[2]} = 0.3$	$p_{[2]} = 0.5$	$p_{[2]} = 0.7$	$p_{[2]} = 0.9$
20	0.000	0.000	0.000	0.001	0.135
21	0.000	0.000	0.000	0.004	0.064
22	0.000	0.000	0.000	0.008	0.061
23	0.000	0.000	0.000	0.013	0.059
24	0.000	0.000	0.001	0.019	0.057
25	0.000	0.000	0.003	0.026	0.055
26	0.000	0.000	0.005	0.033	0.053
27	0.000	0.000	0.010	0.041	0.051
28	0.000	0.001	0.017	0.049	0.050
29	0.000	0.003	0.026	0.057	0.048
30	0.000	0.006	0.037	0.064	0.046
31	0.000	0.013	0.051	0.070	0.044
32	0.000	0.026	0.066	0.074	0.042
33	0.001	0.045	0.081	0.078	0.040
34	0.006	0.071	0.095	0.081	0.038
35	0.020	0.102	0.108	0.081	0.036
36	0.058	0.135	0.118	0.081	0.034
37	0.137	0.167	0.125	0.078	0.031
38	0.265	0.196	0.129	0.074	0.029
39	0.514	0.234	0.129	0.068	0.026
$E\{N\}$	38.172	36.644	34.985	32.562	27.529

Table 4.7

$E\{N\}$ for p^* when $k = 2$ and $p_{[2]} = p_{[1]}$

with $n = 5(5)100$

n	$p_{[2]}$				
	0.1	0.3	0.5	0.7	0.9
5	8.809	8.219	7.539	6.759	5.777
10	18.589	17.585	16.476	14.972	12.586
15	28.371	27.077	25.666	23.648	19.883
20	38.172	36.644	34.985	32.562	27.529
25	47.990	46.260	44.386	41.619	35.449
30	57.823	55.912	53.845	50.773	43.586
35	67.667	65.591	63.348	59.999	51.899
40	77.521	75.292	72.886	69.282	60.352
45	87.382	85.011	82.452	78.609	68.919
50	97.251	94.744	92.041	87.974	77.580
55	107.125	104.491	101.651	97.372	86.320
60	117.005	114.249	111.278	106.796	95.127
65	126.889	124.016	120.920	116.245	103.990
70	136.778	133.792	130.576	125.715	112.903
75	146.670	143.576	140.244	135.204	121.859
80	156.566	153.368	149.923	144.710	130.853
85	166.464	163.165	159.612	154.232	139.882
90	176.366	172.968	169.310	163.768	148.942
95	186.270	182.777	179.016	173.316	158.030
100	196.176	192.590	188.730	182.877	167.143

Table 4.8

Distribution of N , and $E\{N\}$ for p^* when

$$k = 2, n = 5 \text{ and } p_{[2]} - 0.2 = p_{[1]}$$

a	$P\{N = a\}$ for				
	$p_{[2]} = 0.2$	$p_{[2]} = 0.4$	$p_{[2]} = 0.5$	$p_{[2]} = 0.7$	$p_{[2]} = 0.9$
5	0.001	0.005	0.041	0.169	0.490
6	0.008	0.029	0.124	0.234	0.191
7	0.059	0.122	0.231	0.225	0.139
8	0.236	0.303	0.295	0.202	0.103
9	0.696	0.540	0.310	0.170	0.077
$E\{N\}$	8.618	8.345	7.709	6.970	6.088

Table 4.9

Distribution of N , and $E\{N\}$ for p^* when

$$k = 2, n = 10 \text{ and } p_{[2]} - 0.2 = p_{[1]}$$

a	P{N = a} for				
	$p_{[2]} = 0.2$	$p_{[2]} = 0.3$	$p_{[2]} = 0.5$	$p_{[2]} = 0.7$	$p_{[2]} = 0.9$
10	0.000	0.000	0.001	0.025	0.249
11	0.000	0.000	0.007	0.065	0.160
12	0.000	0.001	0.024	0.095	0.129
13	0.001	0.007	0.055	0.116	0.106
14	0.006	0.026	0.096	0.128	0.088
15	0.027	0.071	0.135	0.130	0.074
16	0.089	0.142	0.162	0.126	0.062
17	0.204	0.212	0.174	0.117	0.052
18	0.312	0.251	0.174	0.106	0.043
19	0.362	0.289	0.172	0.092	0.036
E{N}	17.875	17.430	16.406	15.018	12.868

Table 4.10

Distribution of N , and $E\{N\}$ for P^* when

$$k = 2, n = 20 \text{ and } p_{[2]} - 0.2 = p_{[1]}$$

a	$P\{N = a\}$ for				
	$p_{[2]} = 0.2$	$p_{[2]} = 0.3$	$p_{[2]} = 0.5$	$p_{[2]} = 0.7$	$p_{[2]} = 0.9$
20	0.000	0.000	0.000	0.001	0.082
21	0.000	0.000	0.000	0.003	0.093
22	0.000	0.000	0.000	0.009	0.089
23	0.000	0.000	0.000	0.017	0.085
24	0.000	0.000	0.001	0.027	0.081
25	0.000	0.000	0.004	0.039	0.075
26	0.000	0.000	0.009	0.051	0.069
27	0.000	0.001	0.018	0.062	0.062
28	0.000	0.002	0.031	0.071	0.056
29	0.000	0.006	0.047	0.077	0.050
30	0.002	0.016	0.064	0.080	0.044
31	0.007	0.034	0.081	0.080	0.039
32	0.022	0.062	0.094	0.078	0.034
33	0.055	0.097	0.102	0.074	0.030
34	0.109	0.130	0.104	0.069	0.026
35	0.175	0.151	0.102	0.064	0.023
36	0.218	0.152	0.095	0.058	0.020
37	0.206	0.134	0.088	0.052	0.017
38	0.138	0.111	0.081	0.047	0.015
39	0.068	0.105	0.078	0.042	0.012
$E\{N\}$	35.987	35.397	33.805	31.321	26.438

Table 4.11A

$E\{N\}$ for p^* when $k = 2$ and $p_{[2]} - 0.2 = p_{[1]}$

with $n = 5(5)100$

n	$p_{[2]}$				
	0.2	0.3	0.5	0.7	0.9
5	8.618	8.345	7.709	6.970	6.088
10	17.875	17.430	16.406	15.018	12.868
15	26.959	26.433	25.114	23.169	19.670
20	35.987	35.397	33.805	31.321	26.438
25	44.996	44.336	42.474	39.454	33.175
30	53.999	53.259	51.125	47.563	39.889
35	63.000	62.171	59.759	55.652	46.589
40	72.000	71.076	68.381	63.723	53.278
45	81.000	79.976	76.992	71.780	59.960
50	90.000	88.872	85.595	79.825	66.638
55	99.000	97.766	94.191	87.861	73.313
60	108.000	106.659	102.782	95.890	79.985
65	117.000	115.550	111.369	103.912	86.656
70	126.000	124.440	119.953	111.930	93.326
75	135.000	133.331	128.534	119.945	99.995
80	144.000	142.220	137.113	127.956	106.663
85	153.000	151.110	145.690	135.965	113.331
90	162.000	159.999	154.267	143.972	119.998
95	171.000	168.888	162.842	151.978	126.665
100	180.000	177.777	171.417	159.982	133.332

Table 4.11B

$P\{CS\}$ for p^* when $k = 2$ and $p_{[2]} - 0.2 = p_{[1]}$

with $n = 5(5)100$

n	$p_{[2]}$				
	0.6	0.7 or 0.5	0.8 or 0.4	0.9 or 0.3	1.0 or 0.2
5	0.7334	0.7374	0.7507	0.7786	0.8362
10	0.8139	0.8188	0.8352	0.8697	0.9463
15	0.8638	0.8688	0.8852	0.9181	0.9824
20	0.8979	0.9027	0.9180	0.9470	0.9942
25	0.9224	0.9268	0.9404	0.9650	0.9981
30	0.9404	0.9443	0.9563	0.9767	0.9994
35	0.9539	0.9574	0.9677	0.9843	0.9998
40	0.9642	0.9672	0.9760	0.9894	0.9999
45	0.9720	0.9746	0.9820	0.9927	
50	0.9781	0.9803	0.9865	0.9950	
55	0.9828	0.9846	0.9898	0.9966	
60	0.9864	0.9880	0.9923	0.9977	
65	0.9893	0.9906	0.9942	0.9984	
70	0.9915	0.9926	0.9956	0.9989	
75	0.9933	0.9942	0.9967	0.9992	
80	0.9946	0.9954	0.9975	0.9995	
85	0.9957	0.9964	0.9981	0.9996	
90	0.9966	0.9972	0.9985	0.9997	
95	0.9973	0.9978	0.9989	0.9998	
100	0.9978	0.9982	0.9991	0.9999	

Table 4.12

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2$) for P^*

when $k = 2, n = 5$ and $p_{[2]} - 0.2 = p_{[1]}$

a	$p_{[1]}=0.0, p_{[2]}=0.2$		$p_{[1]}=0.1, p_{[2]}=0.3$		$p_{[1]}=0.3, p_{[2]}=0.5$	
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$
0	0.0002	0.0000	0.0012	0.0000	0.0156	0.0012
1	0.0059	0.0000	0.0241	0.0003	0.1094	0.0134
2	0.0493	0.0000	0.1060	0.0039	0.1794	0.0374
3	0.1971	0.0000	0.2368	0.0258	0.1958	0.0673
4	0.5427	0.2728	0.3961	0.3002	0.2482	0.2912
5	0.2048	0.7272	0.2358	0.6698	0.2517	0.5895
$E\{N_{(1)}\}$	3.891	---	3.710	---	3.307	---
$E\{N_{(2)}\}$	---	4.727	---	4.635	---	4.402

Table 4.12 (continued)

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2$) for P^*

when $k = 2$, $n = 5$ and $p_{[2]} - 0.2 = p_{[1]}$

a	$p_{[1]}=0.5, p_{[2]}=0.7$		$p_{[1]}=0.7, p_{[2]}=0.9$		$p_{[1]}=0.8, p_{[2]}=1.0$	
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$
0	0.0840	0.0156	0.2952	0.0840	0.5000	0.1638
1	0.2041	0.0469	0.1771	0.0408	0.1000	0.0000
2	0.1664	0.0544	0.1194	0.0335	0.0800	0.0000
3	0.1304	0.0577	0.0834	0.0273	0.0640	0.0000
4	0.1821	0.3075	0.1219	0.3418	0.0512	0.3362
5	0.2330	0.5179	0.2028	0.4726	0.2048	0.5000
$E\{N_{(1)}\}$	2.821	---	2.168	---	1.681	---
$E\{N_{(2)}\}$	---	4.148	---	3.920	---	3.845

Table 4.13

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2$) for p^*

when $k = 2$, $n = 10$ and $p_{[2]} - 0.2 = p_{[1]}$

a	$p_{[1]}=0.0, p_{[2]}=0.2$		$p_{[1]}=0.1, p_{[2]}=0.3$		$p_{[1]}=0.3, p_{[2]}=0.5$	
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$
0	0.0000	0.0000	0.0000	0.0000	0.0005	0.0000
1	0.0000	0.0000	0.0001	0.0000	0.0068	0.0001
2	0.0001	0.0000	0.0012	0.0000	0.0234	0.0004
3	0.0008	0.0000	0.0068	0.0000	0.0533	0.0015
4	0.0055	0.0000	0.0261	0.0000	0.0907	0.0042
5	0.0264	0.0000	0.0706	0.0003	0.1232	0.0095
6	0.0879	0.0000	0.1387	0.0017	0.1398	0.0180
7	0.2007	0.0000	0.1995	0.0074	0.1364	0.0290
8	0.2995	0.0000	0.2099	0.0231	0.1157	0.0402
9	0.3121	0.0895	0.2094	0.1430	0.1323	0.1490
10	0.0671	0.9105	0.1378	0.8244	0.1779	0.7482
$E\{N_{(1)}\}$	7.964	---	7.650	---	6.881	---
$E\{N_{(2)}\}$	---	9.911	---	9.780	---	9.525

Table 4.13 (continued)

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2$) for P^*

when $k = 2$, $n = 10$ and $p_{[2]} - 0.2 = p_{[1]}$

a	$p_{[1]}=0.5, p_{[2]}=0.7$		$p_{[1]}=0.7, p_{[2]}=0.9$		$p_{[1]}=0.8, p_{[2]}=1.0$	
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$
0	0.0141	0.0005	0.1743	0.0141	0.5000	0.0537
1	0.0646	0.0029	0.1627	0.0129	0.1000	0.0000
2	0.0885	0.0058	0.1247	0.0129	0.0800	0.0000
3	0.1055	0.0097	0.0981	0.0130	0.0640	0.0000
4	0.1116	0.0144	0.0783	0.0131	0.0512	0.0000
5	0.1080	0.0193	0.0627	0.0130	0.0410	0.0000
6	0.0974	0.0241	0.0503	0.0126	0.0328	0.0000
7	0.0825	0.0278	0.0403	0.0120	0.0262	0.0000
8	0.0657	0.0295	0.0322	0.0108	0.0210	0.0000
9	0.0929	0.1750	0.0589	0.3070	0.0168	0.4463
10	0.1694	0.6909	0.1173	0.5786	0.0671	0.5000
$E\{N_{(1)}\}$	5.761	---	3.878	---	2.232	---
$E\{N_{(2)}\}$	---	9.257	---	8.990	---	9.017

Table 4.14

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2$) for p^*

when $k = 2$, $n = 20$ and $p_{[2]} - 0.2 = p_{[1]}$

a	$p_{[1]}=0.0, p_{[2]}=0.2$		$p_{[1]}=0.1, p_{[2]}=0.3$		$p_{[1]}=0.3, p_{[2]}=0.5$	
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
2	0.0000	0.0000	0.0000	0.0000	0.0001	0.0000
3	0.0000	0.0000	0.0000	0.0000	0.0004	0.0000
4	0.0000	0.0000	0.0000	0.0000	0.0015	0.0000
5	0.0000	0.0000	0.0000	0.0000	0.0040	0.0000
6	0.0000	0.0000	0.0001	0.0000	0.0092	0.0000
7	0.0000	0.0000	0.0006	0.0000	0.0179	0.0000
8	0.0001	0.0000	0.0020	0.0000	0.0306	0.0001
9	0.0005	0.0000	0.0061	0.0000	0.0466	0.0002
10	0.0020	0.0000	0.0156	0.0000	0.0639	0.0004
11	0.0074	0.0000	0.0336	0.0000	0.0801	0.0008
12	0.0222	0.0000	0.0616	0.0000	0.0925	0.0016
13	0.0545	0.0000	0.0967	0.0000	0.0991	0.0028
14	0.1091	0.0000	0.1300	0.0002	0.0993	0.0046
15	0.1746	0.0000	0.1502	0.0006	0.0934	0.0072
16	0.2182	0.0000	0.1493	0.0018	0.0828	0.0107
17	0.2053	0.0000	0.1280	0.0047	0.0692	0.0147
18	0.1366	0.0000	0.0943	0.0104	0.0539	0.0188
19	0.0623	0.0096	0.0762	0.0504	0.0594	0.0650
20	0.0072	0.9904	0.0557	0.9318	0.0962	0.8731
$E\{N_{(1)}\}$	15.996	---	15.494	---	14.104	---
$E\{N_{(2)}\}$	---	19.990	---	19.903	---	19.700

Table 4.14 (continued)

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2$) for P^*

when $k = 2$, $n = 20$ and $p_{[2]} - 0.2 = p_{[1]}$

a	$p_{[1]}=0.5, p_{[2]}=0.7$		$p_{[1]}=0.7, p_{[2]}=0.9$		$p_{[1]}=0.8, p_{[2]}=1.0$	
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$
0	0.0004	0.0000	0.0608	0.0004	0.5000	0.0058
1	0.0035	0.0000	0.0973	0.0007	0.1000	0.0000
2	0.0087	0.0000	0.0915	0.0009	0.0800	0.0000
3	0.0167	0.0001	0.0861	0.0011	0.0640	0.0000
4	0.0272	0.0001	0.0803	0.0013	0.0512	0.0000
5	0.0389	0.0002	0.0740	0.0016	0.0410	0.0000
6	0.0507	0.0005	0.0673	0.0018	0.0328	0.0000
7	0.0612	0.0008	0.0605	0.0021	0.0262	0.0000
8	0.0694	0.0012	0.0537	0.0024	0.0210	0.0000
9	0.0747	0.0018	0.0473	0.0027	0.0168	0.0000
10	0.0768	0.0026	0.0412	0.0030	0.0134	0.0000
11	0.0759	0.0037	0.0356	0.0032	0.0107	0.0000
12	0.0725	0.0049	0.0305	0.0035	0.0086	0.0000
13	0.0670	0.0063	0.0260	0.0037	0.0069	0.0000
14	0.0602	0.0079	0.0220	0.0039	0.0055	0.0000
15	0.0526	0.0095	0.0185	0.0040	0.0044	0.0000
16	0.0446	0.0111	0.0154	0.0040	0.0035	0.0000
17	0.0366	0.0125	0.0128	0.0039	0.0028	0.0000
18	0.0290	0.0131	0.0106	0.0036	0.0023	0.0000
19	0.0406	0.0722	0.0192	0.1760	0.0018	0.4942
20	0.0930	0.8516	0.0496	0.7764	0.0072	0.5000
$E\{N_{(1)}\}$	11.797	---	7.015	---	2.471	---
$E\{N_{(2)}\}$	---	19.525	---	19.423	---	19.390

Table 4.15

$E\{N_{(i)}\}$ ($i = 1, 2$) for p^* when $k = 2$ and $p_{[2]}^{-0.2} = p_{[1]}$

with $n = 5(5)100$

n	$p_{[1]}=0.0, p_{[2]}=0.2$		$p_{[1]}=0.1, p_{[2]}=0.3$		$p_{[1]}=0.3, p_{[2]}=0.5$	
	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$
5	3.891	4.727	3.710	4.635	3.307	4.402
10	7.964	9.911	7.650	9.780	6.881	9.525
15	11.988	14.971	11.578	14.855	10.492	14.622
20	15.996	19.990	15.494	19.903	14.104	19.700
25	19.999	24.997	19.402	24.934	17.711	24.763
30	24.000	29.999	23.304	29.955	21.312	29.813
35	28.000	35.000	27.202	34.969	24.907	34.852
40	32.000	40.000	31.097	39.979	28.497	39.883
45	36.000	45.000	34.990	44.985	32.084	44.908
50	40.000	50.000	38.882	49.990	35.668	49.927
55	44.000	55.000	42.773	54.993	39.249	54.942
60	48.000	60.000	46.663	59.995	42.828	59.954
65	52.000	65.000	50.553	64.997	46.405	64.964
70	56.000	70.000	54.443	69.998	49.981	69.971
75	60.000	75.000	58.332	74.998	53.557	74.977
80	64.000	80.000	62.221	79.999	57.131	79.982
85	68.000	85.000	66.111	84.999	60.705	84.986
90	72.000	90.000	70.000	89.999	64.278	89.989
95	76.000	95.000	73.889	95.000	67.851	94.991
100	80.000	100.000	77.778	100.000	71.424	99.993

Table 4.15 (continued)

$E\{N_{(i)}\}$ ($i = 1, 2$) for p^* when $k = 2$ and $p_{[2]}^{-0.2} = p_{[1]}$

with $n = 5(5)100$

n	$p_{[1]}=0.5, p_{[2]}=0.7$		$p_{[1]}=0.7, p_{[2]}=0.9$		$p_{[1]}=0.8, p_{[2]}=1.0$	
	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$
5	2.821	4.148	2.168	3.920	1.681	3.845
10	5.761	9.257	3.878	8.990	2.232	9.017
15	8.766	14.402	5.457	14.212	2.412	14.254
20	11.797	19.525	7.015	19.423	2.471	19.390
25	14.831	24.623	8.587	24.588	2.491	24.455
30	17.862	29.701	10.180	29.709	2.497	29.482
35	20.888	34.764	11.792	34.797	2.499	34.493
40	23.910	39.813	13.420	39.858	2.500	39.497
45	26.928	44.852	15.059	44.901	2.500	44.499
50	29.943	49.883	16.707	49.931	2.500	49.500
55	32.954	54.907	18.361	54.952	2.500	54.500
60	35.963	59.926	20.019	59.967	2.500	59.500
65	38.971	64.942	21.679	64.977	2.500	64.500
70	41.997	69.954	23.342	69.984	2.500	69.500
75	44.982	74.963	25.006	74.989	2.500	74.500
80	47.985	79.971	26.671	79.992	2.500	79.500
85	50.988	84.977	28.336	84.995	2.500	84.500
90	53.991	89.982	30.002	89.996	2.500	89.500
95	56.993	94.985	31.668	94.997	2.500	94.500
100	59.994	99.988	33.334	99.998	2.500	99.500

Table 4.16

Distribution of N , and $E\{N\}$ for p^* when

$k = 3, n = 5$ and $p_{[3]} = p_{[2]} = p_{[1]}$

a	$P\{N = a\}$ for				
	$p_{[3]} = 0.1$	$p_{[3]} = 0.3$	$p_{[3]} = 0.5$	$p_{[3]} = 0.7$	$p_{[3]} = 0.9$
5	0.0000	0.0024	0.0313	0.1681	0.5905
6	0.0001	0.0057	0.0313	0.0720	0.0656
7	0.0009	0.0362	0.1172	0.1433	0.0728
8	0.0008	0.0235	0.0977	0.1455	0.0751
9	0.0147	0.1026	0.1289	0.1360	0.0743
10	0.0097	0.1058	0.1514	0.1241	0.0710
11	0.1099	0.1807	0.1611	0.0997	0.0239
12	0.0866	0.2078	0.1409	0.0620	0.0149
13	0.3968	0.2103	0.0981	0.0354	0.0084
14	0.3805	0.1249	0.0422	0.0138	0.0035
$E\{N\}$	12.976	11.433	9.949	8.400	6.435

Table 4.17

Distribution of N , and $E\{N\}$ for P^* when

$k = 3, n = 7$ and $p_{[3]} = p_{[2]} = p_{[1]}$

a	$P\{N = a\}$ for				
	$p_{[3]} = 0.1$	$p_{[3]} = 0.3$	$p_{[3]} = 0.5$	$p_{[3]} = 0.7$	$p_{[3]} = 0.9$
7	0.0000	0.0002	0.0078	0.0824	0.4783
8	0.0000	0.0005	0.0078	0.0353	0.0531
9	0.0000	0.0048	0.0410	0.0893	0.0622
10	0.0000	0.0030	0.0361	0.0980	0.0665
11	0.0003	0.0191	0.0571	0.0987	0.0680
12	0.0002	0.0210	0.0759	0.0983	0.0669
13	0.0040	0.0493	0.0938	0.0996	0.0645
14	0.0026	0.0681	0.1097	0.0994	0.0609
15	0.0300	0.1033	0.1240	0.0944	0.0290
16	0.0245	0.1368	0.1298	0.0759	0.0203
17	0.1367	0.1680	0.1229	0.0575	0.0139
18	0.1313	0.1811	0.0986	0.0388	0.0089
19	0.3546	0.1571	0.0665	0.0231	0.0052
20	0.3157	0.0876	0.0290	0.0094	0.0022
$E\{N\}$	18.676	16.728	14.785	12.527	9.540

Table 4.18

$E\{N\}$ for p^* when $k = 3$ and $p_{[3]} = p_{[2]} = p_{[1]}$

with $n = 2(2)40$

n	$p_{[3]}$				
	0.1	0.3	0.5	0.7	0.9
2	4.554	3.894	3.375	2.862	2.298
4	10.142	8.843	7.641	6.455	4.976
6	15.822	14.065	12.337	10.426	7.959
8	21.536	19.414	17.279	14.692	11.168
10	27.271	24.843	22.371	19.188	15.545
12	33.023	30.329	27.563	23.858	18.059
14	38.791	35.857	32.828	28.661	21.693
16	44.573	41.419	38.149	33.567	25.438
18	50.368	47.009	43.514	38.554	29.285
20	56.173	52.621	48.915	43.606	33.231
22	61.987	58.253	54.348	48.713	37.268
24	67.810	63.902	59.807	53.864	41.392
26	73.640	69.565	65.289	59.055	45.596
28	79.476	75.241	70.791	64.279	49.875
30	85.317	80.929	76.311	69.534	54.223
32	91.164	86.627	81.848	74.814	58.635
34	97.016	92.335	87.400	80.119	63.106
36	102.872	98.051	92.964	85.445	67.632
38	108.731	103.776	98.541	90.791	72.208
40	114.595	109.507	104.130	96.155	76.831

Table 4.19

Distribution of N , and $E\{N\}$ for P^* when

$k = 3, n = 5$ and $p_{[3]} = 0.2 = p_{[2]} = p_{[1]}$

a	$P\{N = a\}$ for				
	$p_{[3]} = 0.2$	$p_{[3]} = 0.3$	$p_{[3]} = 0.5$	$p_{[3]} = 0.7$	$p_{[3]} = 0.9$
5	0.0001	0.0008	0.0120	0.0769	0.3089
6	0.0006	0.0030	0.0204	0.0626	0.1059
7	0.0051	0.0194	0.0873	0.1558	0.1343
8	0.0026	0.0089	0.0546	0.1316	0.1291
9	0.0461	0.0876	0.1336	0.1324	0.1150
10	0.0102	0.0433	0.1391	0.1352	0.0979
11	0.1843	0.2013	0.1682	0.1297	0.0567
12	0.0410	0.1386	0.1737	0.0951	0.0307
13	0.3277	0.2926	0.1416	0.0580	0.0157
14	0.3823	0.2045	0.0697	0.0227	0.0058
$E\{N\}$	12.709	11.995	10.572	9.087	7.444

Table 4.20

Distribution of N , and $E\{N\}$ for P^* when

$k = 3, n = 7$ and $p_{[3]} = 0.2 = p_{[2]} = p_{[1]}$

a	P{N = a} for				
	$p_{[3]} = 0.2$	$p_{[3]} = 0.3$	$p_{[3]} = 0.5$	$p_{[3]} = 0.7$	$p_{[3]} = 0.9$
7	0.0000	0.0001	0.0027	0.0327	0.2143
8	0.0000	0.0003	0.0046	0.0272	0.0771
9	0.0003	0.0025	0.0274	0.0867	0.1092
10	0.0001	0.0009	0.0166	0.0764	0.1106
11	0.0041	0.0161	0.0533	0.0848	0.1027
12	0.0004	0.0069	0.0577	0.0952	0.0915
13	0.0279	0.0539	0.0840	0.1030	0.0815
14	0.0016	0.0325	0.1023	0.1060	0.0722
15	0.1114	0.1154	0.1203	0.1062	0.0544
16	0.0066	0.0901	0.1336	0.0972	0.0363
17	0.2621	0.1768	0.1389	0.0799	0.0240
18	0.0262	0.1635	0.1242	0.0565	0.0147
19	0.3146	0.2012	0.0911	0.0344	0.0082
20	0.2447	0.1398	0.0432	0.0138	0.0032
E{N}	18.014	17.190	15.405	13.342	10.794

Table 4.21A

$E\{N\}$ for p^* when $k = 3$ and $p_{[3]} - 0.2 = p_{[2]} = p_{[1]}$

with $n = 2(2)40$

n	$p_{[3]}$				
	0.2	0.3	0.5	0.7	0.9
2	4.640	4.259	3.672	3.172	2.649
4	10.036	9.397	8.202	7.034	5.798
6	15.367	14.593	12.978	11.194	9.112
8	20.651	19.786	17.849	15.521	12.487
10	25.905	24.972	22.765	19.940	15.889
12	31.139	30.152	27.702	24.411	19.303
14	36.361	35.326	32.648	28.908	22.718
16	41.575	40.494	37.597	33.419	26.129
18	46.784	45.655	42.545	37.935	29.532
20	51.990	50.810	47.490	42.451	32.927
22	57.193	55.960	52.430	46.964	36.314
24	62.396	61.106	57.366	51.473	39.694
26	67.597	66.246	62.297	55.976	43.068
28	72.798	71.384	67.223	60.473	46.435
30	77.999	76.517	72.144	64.964	49.798
32	83.199	81.648	77.061	69.448	53.157
34	88.400	86.776	81.972	73.926	56.513
36	93.600	91.902	86.880	78.398	59.865
38	98.800	97.026	91.783	82.865	63.215
40	104.000	102.149	96.683	87.327	66.563

Table 4.218

$P\{CS\}$ for p^* when $k = 3$ and $p_{[3]} - 0.2 = p_{[2]} = p_{[1]}$

with $n = 2(2)50$

n	$p_{[3]}$				
	0.2	0.3	0.5	0.7	0.9
2	0.5733	0.5376	0.5005	0.4933	0.4968
4	0.7269	0.6355	0.5768	0.5716	0.6074
6	0.8252	0.7022	0.6329	0.6285	0.6819
8	0.8882	0.7531	0.6778	0.6740	0.7373
10	0.9284	0.7935	0.7151	0.7118	0.7808
12	0.9542	0.8263	0.7469	0.7440	0.8159
14	0.9707	0.8532	0.7743	0.7717	0.8447
16	0.9812	0.8755	0.7982	0.7959	0.8685
18	0.9880	0.8942	0.8191	0.8171	0.8883
20	0.9923	0.9099	0.8376	0.8358	0.9050
22	0.9951	0.9231	0.8539	0.8523	0.9190
24	0.9969	0.9342	0.8685	0.8670	0.9309
26	0.9980	0.9437	0.8814	0.8802	0.9409
28	0.9987	0.9518	0.8930	0.8919	0.9494
30	0.9992	0.9586	0.9034	0.9024	0.9567
32	0.9995	0.9645	0.9127	0.9118	0.9629
34	0.9997	0.9695	0.9211	0.9202	0.9681
36	0.9998	0.9738	0.9286	0.9278	0.9726
38	0.9999	0.9774	0.9353	0.9347	0.9765
40		0.9806	0.9414	0.9408	0.9798
42		0.9833	0.9469	0.9464	0.9826
44		0.9856	0.9519	0.9514	0.9850
46		0.9876	0.9563	0.9559	0.9871
48		0.9893	0.9604	0.9600	0.9889
50		0.9907	0.9640	0.9637	0.9904

Table 4.22

Distribution of N , and $E\{N\}$ for P^* when
 $k = 3, n = 5$ and equally-spaced $p_{[1]}, p_{[2]}, p_{[3]}$

a	$P\{N = a\}$ for				
	$p_{[1]} = 0.6$	$p_{[1]} = 0.5$	$p_{[1]} = 0.4$	$p_{[1]} = 0.3$	$p_{[1]} = 0.2$
	$p_{[2]} = 0.6$	$p_{[2]} = 0.6$	$p_{[2]} = 0.6$	$p_{[2]} = 0.6$	$p_{[2]} = 0.6$
	$p_{[3]} = 0.6$	$p_{[3]} = 0.7$	$p_{[3]} = 0.8$	$p_{[3]} = 0.9$	$p_{[3]} = 1.0$
5	0.0778	0.0924	0.1386	0.2236	0.3594
6	0.0518	0.0642	0.1046	0.1832	0.3171
7	0.1431	0.1544	0.1805	0.1961	0.1541
8	0.1337	0.1385	0.1462	0.1378	0.0852
9	0.1348	0.1341	0.1268	0.1013	0.0497
10	0.1388	0.1329	0.1130	0.0765	0.0295
11	0.1351	0.1243	0.0932	0.0482	0.0041
12	0.0990	0.0872	0.0567	0.0218	0.0008
13	0.0613	0.0520	0.0299	0.0089	0.0001
14	0.0245	0.0202	0.0105	0.0026	0.0000
$E\{N\}$	9.188	8.942	8.266	7.313	6.259

Table 4.23

Distribution of N , and $E\{N\}$ for P^* when
 $k = 3$, $n = 7$ and equally-spaced $p[1]$, $p[2]$, $p[3]$

a	$P\{N = a\}$ for				
	$p[1] = 0.6$	$p[1] = 0.5$	$p[1] = 0.4$	$p[1] = 0.3$	$p[1] = 0.2$
	$p[2] = 0.6$	$p[2] = 0.6$	$p[2] = 0.6$	$p[2] = 0.6$	$p[2] = 0.6$
	$p[3] = 0.6$	$p[3] = 0.7$	$p[3] = 0.8$	$p[3] = 0.9$	$p[3] = 1.0$
7	0.0280	0.0394	0.0798	0.1688	0.3427
8	0.0187	0.0278	0.0618	0.1421	0.3104
9	0.0694	0.0865	0.1322	0.1805	0.1527
10	0.0696	0.0821	0.1119	0.1317	0.0849
11	0.0784	0.0872	0.1036	0.1006	0.0496
12	0.0907	0.0955	0.0994	0.0796	0.0295
13	0.1027	0.1032	0.0955	0.0645	0.0177
14	0.1102	0.1063	0.0888	0.0519	0.0106
15	0.1143	0.1059	0.0795	0.0385	0.0015
16	0.1064	0.0941	0.0612	0.0221	0.0003
17	0.0894	0.0754	0.0422	0.0114	0.0001
18	0.0649	0.0524	0.0256	0.0053	0.0000
19	0.0406	0.0316	0.0136	0.0022	0.0000
20	0.0168	0.0126	0.0050	0.0007	0.0000
$E\{N\}$	13.706	13.204	11.886	10.163	8.412

Table 4.24A

$E\{N\}$ for P^* when $k = 3$, $n = 2(2)40$ and
equally-spaced $p_{[1]}, p_{[2]}, p_{[3]}$

n	$p_{[1]} = 0.6$	$p_{[1]} = 0.5$	$p_{[1]} = 0.4$	$p_{[1]} = 0.3$	$p_{[1]} = 0.2$
	$p_{[2]} = 0.6$	$p_{[2]} = 0.6$	$p_{[2]} = 0.6$	$p_{[2]} = 0.6$	$p_{[2]} = 0.6$
	$p_{[3]} = 0.6$	$p_{[3]} = 0.7$	$p_{[3]} = 0.8$	$p_{[3]} = 0.9$	$p_{[3]} = 1.0$
2	3.123	3.098	3.021	2.896	2.722
4	7.054	6.905	6.485	5.863	5.133
6	11.410	11.047	10.069	8.746	7.349
8	16.060	15.405	13.714	11.570	9.456
10	20.906	19.900	17.395	14.358	11.505
12	25.888	24.483	21.096	17.124	13.526
14	30.967	29.125	24.810	19.878	15.535
16	36.120	33.806	28.530	22.626	17.539
18	41.332	38.513	32.253	25.373	19.541
20	46.590	43.240	35.977	28.120	21.541
22	51.888	47.980	39.699	30.870	23.542
24	57.220	52.729	43.420	33.623	25.542
26	62.581	57.485	47.137	36.379	27.542
28	67.967	62.246	50.852	39.139	29.542
30	73.376	67.010	54.563	41.903	31.542
32	78.806	71.777	58.270	44.669	33.542
34	84.254	76.545	61.975	47.439	35.542
36	89.718	81.314	65.676	50.211	37.542
38	95.198	86.083	69.373	52.985	39.542
40	100.692	90.851	73.069	55.761	41.542

Table 4.24B

$P\{CS\}$ for p^* when $k = 3$, $n = 2(2)50$ and
equally-spaced $p_{[1]}, p_{[2]}, p_{[3]}$

n	$p_{[1]} = 0.6$ $p_{[2]} = 0.6$ $p_{[3]} = 0.6$	$p_{[1]} = 0.5$ $p_{[2]} = 0.6$ $p_{[3]} = 0.7$	$p_{[1]} = 0.4$ $p_{[2]} = 0.6$ $p_{[3]} = 0.8$	$p_{[1]} = 0.3$ $p_{[2]} = 0.6$ $p_{[3]} = 0.9$	$p_{[1]} = 0.2$ $p_{[2]} = 0.6$ $p_{[3]} = 1.0$
2	0.3333	0.4488	0.5696	0.6902	0.8048
4	0.3333	0.5051	0.6740	0.8198	0.9345
6	0.3333	0.5460	0.7398	0.8837	0.9766
8	0.3333	0.5790	0.7866	0.9210	0.9916
10	0.3333	0.6070	0.8217	0.9448	0.9970
12	0.3333	0.6313	0.8491	0.9608	0.9989
14	0.3333	0.6529	0.8710	0.9718	0.9996
16	0.3333	0.6721	0.8889	0.9796	0.9999
18	0.3333	0.6895	0.9038	0.9851	
20	0.3333	0.7053	0.9162	0.9891	
22	0.3333	0.7198	0.9268	0.9920	
24	0.3333	0.7332	0.9358	0.9941	
26	0.3333	0.7455	0.9435	0.9956	
28	0.3333	0.7570	0.9502	0.9968	
30	0.3333	0.7676	0.9561	0.9976	
32	0.3333	0.7775	0.9611	0.9982	
34	0.3333	0.7868	0.9656	0.9987	
36	0.3333	0.7955	0.9695	0.9990	
38	0.3333	0.8037	0.9729	0.9993	
40	0.3333	0.8114	0.9759	0.9995	
42	0.3333	0.8187	0.9786	0.9996	
44	0.3333	0.8256	0.9809	0.9997	
46	0.3333	0.8321	0.9830	0.9998	
48	0.3333	0.8383	0.9849	0.9998	
50	0.3333	0.8441	0.9865	0.9999	

Table 4.25

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2, 3$) for P^*

when $k = 3$, $n = 5$ and $(p_{[1]}, p_{[2]}, p_{[3]}) = (0.5, 0.6, 0.7)$ and $(0.4, 0.6, 0.8)$

a	$p_{[1]}=0.5, p_{[2]}=0.6, p_{[3]}=0.7$			$p_{[1]}=0.4, p_{[2]}=0.6, p_{[3]}=0.8$		
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(3)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(3)}=a\}$
0	0.1186	0.0979	0.0537	0.1942	0.1678	0.0437
1	0.2656	0.1926	0.1127	0.3577	0.2210	0.0644
2	0.1855	0.1494	0.1008	0.1920	0.1502	0.0581
3	0.1283	0.1156	0.0881	0.1068	0.1062	0.0519
4	0.1181	0.1475	0.1867	0.0703	0.1195	0.2117
5	0.1839	0.2970	0.4580	0.0791	0.2352	0.5702
$E\{N_{(1)}\}$	2.414	---	---	1.738	---	---
$E\{N_{(2)}\}$	---	2.913	---	---	2.494	---
$E\{N_{(3)}\}$	---	---	3.615	---	---	4.034

Table 4.25 (continued)

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2, 3$) for p^*
 when $k = 3$, $n = 5$ and $(p_{[1]}, p_{[2]}, p_{[3]}) = (0.3, 0.6, 0.9)$ and $(0.2, 0.6, 1.0)$

a	$p_{[1]}=0.3, p_{[2]}=0.6, p_{[3]}=0.9$			$p_{[1]}=0.2, p_{[2]}=0.6, p_{[3]}=1.0$		
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(3)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(3)}=a\}$
0	0.3188	0.2960	0.0400	0.5130	0.5001	0.0390
1	0.4215	0.2372	0.0290	0.3896	0.2000	0.0000
2	0.1517	0.1429	0.0251	0.0779	0.1200	0.0000
3	0.0606	0.0910	0.0214	0.0156	0.0720	0.0000
4	0.0274	0.0823	0.2470	0.0031	0.0432	0.3073
5	0.0199	0.1507	0.6374	0.0008	0.0648	0.6537
$E\{N_{(1)}\}$	1.116	---	---	0.609	---	---
$E\{N_{(2)}\}$	---	1.879	---	---	1.153	---
$E\{N_{(3)}\}$	---	---	4.319	---	---	4.498

Table 4.26

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2, 3$) for p^*

when $k = 3$, $n = 7$ and $(p_{[1]}, p_{[2]}, p_{[3]}) = (0.5, 0.6, 0.7)$ and $(0.4, 0.6, 0.8)$

a	$p_{[1]}=0.5, p_{[2]}=0.6, p_{[3]}=0.7$			$p_{[1]}=0.4, p_{[2]}=0.6, p_{[3]}=0.8$		
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(3)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(3)}=a\}$
0	0.0544	0.0449	0.0178	0.1169	0.1056	0.0148
1	0.1800	0.1229	0.0544	0.3004	0.1811	0.0299
2	0.1727	0.1232	0.0640	0.2123	0.1443	0.0328
3	0.1495	0.1168	0.0700	0.1424	0.1167	0.0353
4	0.1180	0.1032	0.0707	0.0895	0.0926	0.0362
5	0.0867	0.0853	0.0659	0.0536	0.0713	0.0347
6	0.0809	0.1093	0.1456	0.0364	0.0810	0.1685
7	0.1578	0.2944	0.5115	0.0484	0.2074	0.6477
$E\{N_{(1)}\}$	3.469	---	---	2.336	---	---
$E\{N_{(2)}\}$	---	4.276	---	---	3.485	---
$E\{N_{(3)}\}$	---	---	5.459	---	---	6.065

Table 4.26 (continued)

Distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($i = 1, 2, 3$) for P^*

when $k = 3$, $n = 7$ and $(p_{[1]}, p_{[2]}, p_{[3]}) = (0.3, 0.6, 0.9)$ and $(0.2, 0.6, 1.0)$

a	$p_{[1]}=0.3, p_{[2]}=0.6, p_{[3]}=0.9$			$p_{[1]}=0.2, p_{[2]}=0.6, p_{[3]}=1.0$		
	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(3)}=a\}$	$P\{N_{(1)}=a\}$	$P\{N_{(2)}=a\}$	$P\{N_{(3)}=a\}$
0	0.2487	0.2392	0.0141	0.5047	0.5000	0.0140
1	0.4209	0.2341	0.0138	0.3963	0.2000	0.0000
2	0.1812	0.1513	0.0135	0.0793	0.1200	0.0000
3	0.0817	0.1024	0.0132	0.0159	0.0720	0.0000
4	0.0367	0.0710	0.0127	0.0032	0.0432	0.0000
5	0.0164	0.0499	0.0117	0.0006	0.0259	0.0000
6	0.0079	0.0475	0.2180	0.0001	0.0156	0.3240
7	0.0065	0.1047	0.7029	0.0000	0.0233	0.6620
$E\{N_{(1)}\}$	1.350	---	---	0.619	---	---
$E\{N_{(2)}\}$	---	2.395	---	---	1.215	---
$E\{N_{(3)}\}$	---	---	6.418	---	---	6.578

Table 4.27

$E\{N_{(i)}\}$ ($i = 1, 2, 3$) for p^* when $k = 3$ and

$(p_{[1]}, p_{[2]}, p_{[3]}) = (0.5, 0.6, 0.7)$ and $(0.4, 0.6, 0.8)$ with $n = 2(2)40$

n	$p_{[1]}=0.5, p_{[2]}=0.6, p_{[3]}=0.7$			$p_{[1]}=0.4, p_{[2]}=0.6, p_{[3]}=0.8$		
	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$	$E\{N_{(3)}\}$	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$	$E\{N_{(3)}\}$
2	0.923	1.025	1.149	0.798	0.977	1.247
4	1.905	2.259	2.741	1.438	1.993	3.054
6	2.935	3.587	4.525	2.037	2.991	5.041
8	4.015	4.977	6.412	2.639	3.977	7.098
10	5.136	6.409	8.354	3.258	4.959	9.178
12	6.287	7.868	10.329	3.894	5.943	11.260
14	7.457	9.346	12.322	4.542	6.931	13.337
16	8.642	10.837	14.327	5.200	7.922	15.407
18	9.837	12.336	16.340	5.865	8.918	17.470
20	11.038	13.843	18.359	6.535	9.916	19.526
22	12.244	15.354	20.382	7.207	10.917	21.575
24	13.453	16.868	22.408	7.882	11.919	23.619
26	14.665	18.384	24.436	8.557	12.922	25.659
28	15.879	19.902	26.465	9.232	13.926	27.694
30	17.094	21.421	28.495	9.907	14.931	29.725
32	18.310	22.941	30.525	10.582	15.935	31.753
34	19.527	24.462	32.556	11.256	16.940	33.778
36	20.744	25.983	34.587	11.930	17.945	35.800
38	21.961	27.504	36.618	12.604	18.949	37.821
40	23.178	29.025	38.648	13.277	19.953	39.839

Table 4.27 (continued)

$E\{N_{(i)}\}$ ($i = 1, 2, 3$) for P^* when $k = 3$ and

$(p_{[1]}, p_{[2]}, p_{[3]}) = (0.3, 0.6, 0.9)$ and $(0.2, 0.6, 1.0)$ with $n = 2(2)40$

n	$p_{[1]}=0.3, p_{[2]}=0.6, p_{[3]}=0.9$			$p_{[1]}=0.2, p_{[2]}=0.6, p_{[3]}=1.0$		
	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$	$E\{N_{(3)}\}$	$E\{N_{(1)}\}$	$E\{N_{(2)}\}$	$E\{N_{(3)}\}$
2	0.665	0.897	1.333	0.528	0.789	1.405
4	0.989	1.589	3.285	0.597	1.087	3.448
6	1.235	2.145	5.366	0.615	1.192	5.542
8	1.465	2.632	7.472	0.621	1.229	7.605
10	1.698	3.086	9.574	0.624	1.242	9.638
12	1.937	3.526	11.662	0.625	1.247	11.654
14	2.183	3.961	13.734	0.625	1.249	13.661
16	2.436	4.399	15.792	0.625	1.250	15.665
18	2.694	4.841	17.837	0.625	1.250	17.666
20	2.958	5.289	19.873	0.625	1.250	19.666
22	3.225	5.744	21.901	0.625	1.250	21.667
24	3.496	6.204	23.923	0.625	1.250	23.667
26	3.769	6.671	25.940	0.625	1.250	25.667
28	4.045	7.142	27.953	0.625	1.250	27.667
30	4.322	7.618	29.963	0.625	1.250	29.667
32	4.601	8.097	31.971	0.625	1.250	31.667
34	4.881	8.580	33.977	0.625	1.250	33.667
36	5.162	9.066	35.982	0.625	1.250	35.667
38	5.445	9.554	37.986	0.625	1.250	37.667
40	5.727	10.045	39.989	0.625	1.250	39.667

5. Discussion of performance characteristic results

In the next three subsections we discuss three related performance characteristics of P^* , specifically, the distribution of N , and $E\{N\}$ in Section 5.1, the distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($1 \leq i \leq k$) in Section 5.2, and the $P\{CS\}$ in Section 5.3. Each characteristic is important in its own right, e.g., $E\{N\}$ can be compared directly with kn , the total number of observations required by the single-stage procedure of Sobel and Huyett, since both achieve the same $P\{CS\}$ uniformly in (p_1, \dots, p_k) ; this comparison can be made without knowledge of the achieved $P\{CS\}$. However, an increase in the $P\{CS\}$ is purchased at the cost of an increase in $E\{N\}$, and therefore it sometimes may be important to know the $P\{CS\}$ that actually would be achieved for particular (p_1, \dots, p_k) when the single-stage procedure (or P^*) is used. Similar trade-offs arise when we consider the distribution of $N_{(i)}$, and $E\{N_{(i)}\}$ ($1 \leq i \leq k$) along with the $P\{CS\}$. For example, $E\{N_{(1)}\}$ is important in clinical trials since in that context it represents the expected number of patients subjected to the least effective treatment, and one would clearly seek to make this quantity small for given $P\{CS\}$. Since the performance characteristics $E\{N\}$, $E\{N_{(i)}\}$ ($1 \leq i \leq k$) and $P\{CS\}$ interplay, we study them together.

5.1 Distribution of N , and $E\{N\}$

Tables 4.4, 4.5 and 4.6 show for $k = 2$ and $n = 5, 10$ and 20 , respectively, how the distribution of N , and $E\{N\}$ react to changes in the common value of $p_{[2]} = p_{[1]}$. Tables 4.16 and 4.17 give analogous results for $k = 3$ and $n = 5$ and 7 , respectively, when the common value of $p_{[3]} = p_{[1]}$ changes. It is to be recalled that $P\{N = n\} \rightarrow 1$ as $p_{[1]} \rightarrow 1$ and $P\{N = kn-1\} \rightarrow 1$ as $p_{[k]} \rightarrow 0$; the approach to these limiting values is

already clearly evident for $p_{[1]} = 0.9$ and $p_{[k]} = 0.1$, respectively. The $E\{N\}$ -values are to be compared with kn , the total number of observations required by the single-stage procedure which achieves the same $P\{CS\}$ as p^* uniformly in (p_1, \dots, p_k) . Clearly the saving in $E\{N\}$ resulting from using p^* in place of the single-stage procedure can be substantial, particularly for $p_{[1]}$ close to unity. Tables 4.7 and 4.18 show how these savings change with increasing n for $k = 2$ and 3 , respectively. For example, when $k = 2$ and $p_{[1]} = p_{[2]} = 0.9$ the absolute saving in $E\{N\}$ increases from $2(5) - 5.777 = 4.223$ to $2(40) - 60.352 = 19.648$ as n increases from 5 to 40 ; when $k = 3$ and $p_{[1]} = p_{[2]} = p_{[3]} = 0.9$ the absolute saving in $E\{N\}$ increases from $3(5) - 6.435 = 8.565$ to $3(40) - 76.831 = 43.169$ as n increases from 5 to 40 . The corresponding relative savings $(kn - E\{N\})/kn$ decrease with increasing n for both $k = 2$ and $k = 3$.

The results in Table 4.7 should also be viewed in the light of Theorem 3.2. Since $p_1 + p_2 \geq 1$ for $p_{[2]} = 0.5, 0.7, 0.9$ of Table 4.7, we know that $E\{N|(p_1, p_2)\}$ cannot be decreased for these (p_1, p_2) for any sampling rule R in the class described at the start of Section 3.1. Also, since $p_1 + p_2 \leq 1$ for $p_{[2]} = 0.5, 0.3$, and 0.1 of Table 4.7, we know that had \bar{R}^* been used in place of R^* for these latter $p_{[2]}$ -values, then $E\{N|(p_1, p_2)\}$ would have been the same as that obtained for $p_{[2]} = 0.5, 0.7$, and 0.9 , respectively, and \bar{R}^* would have been optimal for $p_{[2]} = 0.5, 0.3$, and 0.1 in the same sense that R^* was optimal for $0.5, 0.7$, and 0.9 .

Tables 4.8, 4.9 and 4.10 show for $k = 2$ and $n = 5, 10$ and 20 , respectively, how the distribution of N , and $E\{N\}$ react to changes in $p_{[2]}$ when $p_{[2]} - 0.2 = p_{[1]}$. Tables 4.19 and 4.20 give analogous results for $k = 3$ and $n = 5$ and 7 , respectively, when the value of $p_{[3]} - 0.2 = p_{[2]} = p_{[1]}$ changes.

The results in Tables 4.8, 4.9, 4.10 and 4.11A should be compared with those in Tables 4.4, 4.5, 4.6 and 4.7, respectively, and the results in Tables 4.19, 4.20 and 4.21A with those in Tables 4.16, 4.17 and 4.18, respectively. Such comparisons show (for the k - and p -values considered) how much $E\{N\}$ decreases as the configuration of the $p_{[i]}$ ($1 \leq i \leq k$) becomes "more favorable" (by 0.2) to the experimenter, e.g., compare the results in the $p_{[2]} = 0.9$ ($p_{[3]} = 0.9$) column of Table 4.11A (Table 4.21A) with the corresponding results in the $p_{[2]} = 0.7$ ($p_{[3]} = 0.7$) column of Table 4.7 (Table 4.18). This type of phenomenon is exhibited even more strikingly for $k = 3$ in Tables 4.22, 4.23 and 4.24A.

Theorem 3.2 is, of course, also relevant to the results in Table 4.11A, as is Conjecture 3.1 to the results in Tables 4.18, 4.21A and 4.24A. Until Conjecture 3.1 can be resolved analytically, the authors would be very much interested in learning of any computational results (if such exist) which might indicate that Conjecture 3.1 is false.

5.2 Distribution of $N_{(i)}$ and $E\{N_{(i)}\}$ ($1 \leq i \leq k$)

As remarked at the outset of Section 5, the distribution of $N_{(i)}$ and $E\{N_{(i)}\}$ ($1 \leq i \leq k$) are of interest in various areas of application, e.g., $E\{N_{(1)}\}$ is important in clinical trials where p_i denotes the probability of "success" using treatment i ($1 \leq i \leq k$), and $E\{N_{(1)}\}$ denotes the expected number of patients subjected to the least effective treatment. One would hope that $E\{N_{(1)}\}$ for P^* would be small relative to n if $p_{[1]} \ll p_{[2]}$. Table 4.15 provides numerical evidence to this effect for $k = 2$ while Table 4.27 provides even more striking evidence for $k = 3$. We note from both tables that $E\{N_{(1)}\}/n$ is a decreasing function of n . A comparison of corresponding results in Tables 4.15 and 4.11A also shows for $k = 2$ that

$E\{N_{(1)}\}/E\{N\}$ is a decreasing function of n ; the same conclusion is reached for $k = 3$ from the results in Table 4.27 where $E\{N\} = \sum_{i=1}^3 E\{N_{(i)}\}$. Thus p^* does indeed behave as one would hope for $p_{[1]} \ll p_{[2]}$, and for fixed k and $p = (p_1, \dots, p_k)$ its performance improves in terms of $E\{N_{(1)}\}/kn$ and $E\{N_{(1)}\}/E\{N\}$ as n increases.

We also point out that Theorem 3.3 is relevant to the $E\{N_{(1)}\}$ results in the columns headed $p_{[2]} = 0.5, 0.7, 0.9, 1.0$ in Table 4.15, i.e., the $E\{N_{(1)}\}$ results given in these columns cannot be decreased if one restricts consideration to sampling rules R (described at the outset of Section 3.1). However, Theorem 3.3 is specific to the case $k = 2$, and at the present time we do not have an analogous theorem for $k \geq 3$.

Remark 5.1: We were surprised to note in Table 4.13 which is for $k = 2$ with $p_{[2]} = p_{[1]} + 0.2$ that when $n = 10$, $E\{N_{(2)}\}$ is a strictly decreasing function of $p_{[2]}$ (as one might expect) as $p_{[2]}$ increases from 0.2 to 0.9, but $E\{N_{(2)}\}$ for $p_{[2]} = 0.9$ is less than $E\{N_{(2)}\}$ for $p_{[2]} = 1.0$; in additional calculations not presented herein this same phenomenon prevailed for all n ($9 \leq n \leq 18$) but not for $1 \leq n \leq 8$ or for $n = 19, 20$. We are uncertain as to the cause of this behavior. However, we point out that a discontinuity occurs in the distribution of $N_{(2)}$ as $p_{[2]} \rightarrow 1$; for $p_{[1]} \leq p_{[2]} < 1$ the random variable $N_{(2)}$ can assume all of the values $0(1)n$; however, for $p_{[1]} < p_{[2]} = 1$ ($p_{[1]} = p_{[2]} = 1$) we see that $N_{(2)}$ can assume only the values $0, n-1, n$ ($0, n$); finally, for $p_{[1]} = 0$, $p_{[2]} = 1$ we note that $N_{(2)}$ equals $n-1$ or n . Also, presumably there is a differential effect as n and/or $p_{[2]}$ and Δ change.

Remark 5.2: An upper bound for $E\{N_{(i)}\}$ ($i \neq k$) is derived in Appendix D, and its goodness is assessed.

5.3 Probability of correct selection

Table 4.11B shows for $k = 2$ how the $P\{CS\}$ increases with increasing n when $p_{[2]} - 0.2 = p_{[1]}$, and $p_{[2]}$ has one of the selected values in the table. Table 4.21B provides the same information for $k = 3$ when $p_{[3]} - 0.2 = p_{[2]} = p_{[1]}$, and $p_{[3]}$ has one of the selected values. It is to be noted, for $k = 2$ and all n , that $P\{CS\}$ for $(p_{[1]} = a, p_{[2]} = a + \Delta)$ equals $P\{CS\}$ for $(p_{[1]} = 1 - a - \Delta, p_{[2]} = 1 - a)$ for all (a, Δ) with $a > 0, \Delta > 0, a + \Delta < 1$. ($\Delta = 0.2$ in Table 4.11B and $a = 0.4(0.1)0.8$.) However, for $k = 3$ it is only true asymptotically ($n \rightarrow \infty$) that $P\{CS\}$ for $(p_{[1]} = p_{[2]} = a, p_{[3]} = a + \Delta)$ equals $P\{CS\}$ for $(p_{[1]} = p_{[2]} = 1 - a - \Delta, p_{[3]} = 1 - a)$ for all (a, Δ) with $a > 0, \Delta > 0, a + \Delta < 1$; an appreciation of the rates at which these probabilities approach equality as a function of n for selected a and $\Delta = 0.2$ can be obtained by comparing in Table 4.21B the entries in the column headed $p_{[3]} = 0.3 (0.5)$ with those for corresponding n in the column headed $p_{[3]} = 0.9 (0.7)$, i.e., $a = 0.1 (0.3)$. In this connection see Appendix B.

Remark 5.3: We point out that for $k = 2$ and all n , $E\{N\}$ for $(p_{[1]} = a, p_{[2]} = a + \Delta)$ is greater than (less than) $E\{N\}$ for $(p_{[1]} = 1 - a - \Delta, p_{[2]} = 1 - a)$ if $\Delta < 1 - 2a$ ($\Delta > 1 - 2a$); also, for $k = 3$ and all n , $E\{N\}$ for $(p_{[1]} = p_{[2]} = a, p_{[3]} = a + \Delta)$ is greater than (less than) $E\{N\}$ for $(p_{[1]} = p_{[2]} = 1 - a - \Delta, p_{[3]} = 1 - a)$ if $\Delta < 1 - 2a$ ($\Delta > 1 - 2a$), for all (a, Δ) with $a > 0, \Delta > 0, a + \Delta < 1$. This behavior of $E\{N\}$ is unlike that of $P\{CS\}$ as described above. In fact, for $k = 2$ and all n , $E\{N\}$ is a decreasing function of $p_{[2]}$ if $p_{[2]} - \Delta = p_{[1]}$, and for $k = 3$ and all n , $E\{N\}$ is a decreasing function of $p_{[3]}$ if $p_{[3]} - \Delta = p_{[2]} = p_{[1]}$. Tables 4.11A and 4.21A illustrate this phenomenon for $k = 2$ and $k = 3$, respectively, when $\Delta = 0.2$.

Tables 4.11B and 4.21B can be used as in Sobel and Huyett [1957], equation (12), for designing experiments, i.e., for choosing n , to guarantee the indifference-zone probability requirement

$$P\{CS\} \geq P^* \text{ whenever } p_{[k-1]} \leq p_{[k-1]}^* \text{ and } p_{[k]}^* \leq p_{[k]} \quad (5.1)$$

where $\{p_{[k-1]}^*, p_{[k]}^*, P^*\}$ with $0 \leq p_{[k-1]}^* < p_{[k]}^* \leq 1$ and $1/k < P^* < 1$ are specified by the experimenter prior to the start of experimentation. In Table 4.11B (4.21B) one can regard $p_{[2]}$ ($p_{[3]}$) as playing the role of $p_{[2]}^*$ ($p_{[3]}^*$) and $p_{[1]}$ ($p_{[2]}$) as playing the role of $p_{[1]}^*$ ($p_{[2]}^*$) with $p_{[2]}^* - p_{[1]}^*$ ($p_{[3]}^* - p_{[2]}^*$) equal to 0.2. (Note: Our $p_{[i]}$ ($p_{[i]}^*$) is denoted by $p_{[k-i+1]}$ ($p_{[k-i+1]}^*$) ($1 \leq i \leq k$) in Sobel and Huyett.) Thus, e.g., referring to Table 4.21B we see that for $k = 3$ and a specification of $p_{[3]}^* = 0.5$, $p_{[2]}^* = 0.3$, $P^* = 0.95$, a single-stage sample size of approximately $n = 43$ will be required to guarantee the probability requirement (5.1).

Tables 4.11B and 4.21B can also be used in conjunction with Tables 4.11A and 4.21A, respectively. For example, if an experimenter chooses his single-stage sample size n to achieve a $P\{CS\}$ in Table 4.11A or 4.21A using the probability requirement (5.1), then Tables 4.11B and 4.21B give $E\{N\}$ for P^* used with that same n , and the experimenter is assured (from Theorem 3.1) that P^* achieves the same $P\{CS\}$ as the single-stage procedure uniformly in the unknown $p = (p_1, p_2, \dots, p_k)$.

6. References

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Appendix A: Exact formula for $P\{CS|(R_{SS}, T_{SS})\}$ in the configuration

$$P[1] = \dots = P[k-1] = P[k]^{-\Delta}$$

For $\Delta \geq 0$ let

$$P[k] = p \text{ and } P[1] = \dots = P[k-1] = p - \Delta. \quad (A.1)$$

Define

$$a_i(p) = \binom{n}{i} p^i (1-p)^{n-i} \quad (0 \leq i \leq n)$$

and

$$b_i(p) = \sum_{j=0}^{i-1} a_j(p) \quad (1 \leq i \leq n) \text{ and } b_0(p) = 0.$$

Then when the p_i ($1 \leq i \leq k$) are in the configuration (A.1) we have $P\{CS|(R_{SS}, T_{SS})\}$ given by

$$P\{CS\} = \sum_{i=0}^n \frac{a_i(p)}{k a_i(p-\Delta)} [(b_{i+1}(p-\Delta))^k - (b_i(p-\Delta))^k]. \quad (A.2)$$

Equation (A.2) provides an $O(n)$ algorithm for the computation of $P\{CS|(R_{SS}, T_{SS})\}$ for any k given (A.1).

When $p = 1$ equation (A.2) reduces to

$$P\{CS\} = \frac{1}{k(1-\Delta)^n} [1 - (1-(1-\Delta)^n)^k]. \quad (A.3)$$

Appendix B: Normal approximation to $P\{CS|(R_{SS}, T_{SS})\}$ in the configuration

$P[1] = \dots = P[k-1] = P[k]^{-\Delta}$ for large n

When n is large and $P[k] = p$ (say), $P[1] = \dots = P[k-1] = p^{-\Delta}$ we have that $P\{CS|(R_{SS}, T_{SS})\}$ is given approximately (see Sobel and Huyett [1957], equations (A38) and (A39)) by

$$P\{Z_i \geq \frac{-\sqrt{n} \Delta}{\sqrt{p(1-p) + (p-\Delta)(1-p+\Delta)}} \quad (1 \leq i \leq k-1)\} \quad (B.1)$$

where Z_i ($1 \leq i \leq k-1$) have a standard $(k-1)$ -variate equi-correlated normal distribution with

$$\rho = \text{Corr}\{Z_{i_1}, Z_{i_2}\} = \frac{p(1-p)}{p(1-p) + (p-\Delta)(1-p+\Delta)} \quad (B.2)$$

$$(i_1 \neq i_2, 1 \leq i_1, i_2 \leq k-1)$$

The constant $c_{k,p,\rho}$ in the equation

$$P\{Z_i \geq -\frac{c_{k,p,\rho}}{\sqrt{2}} \quad (1 \leq i \leq k-1)\} = P \quad (B.3)$$

has been tabulated for selected (k, P, ρ) . For example, Table 1 of Bechhofer [1954] gives $c_{k,p,\rho}$ for $k = 2(1)10$, $\rho = 1/2$ and a large range of P -values. Table 1 of Gupta, Nagel and Panchapakesan [1973] gives $c_{k,p,\rho}/\sqrt{2}$ for $k = 2(1)51$, a large range of ρ -values, and $P = 0.75, 0.90, 0.95, 0.975$, and

0.99. Using these tabulated c-values it is possible to obtain an excellent large-sample approximation to n for given (k, P, ρ) using the relationship

$$n \sim \frac{p(1-p) + (p-\Delta)(1-p+\Delta)}{\Delta^2} \frac{(c_{k,p,\rho})^2}{2} \quad (B.4)$$

Example B.1: Suppose that $k = 2$. Then (B.3) reduces to a univariate integral. For $P = 0.995$ (say) we find from Table 1 of Bechhofer [1954] that $c_{2,p,\rho} = c_{2,0.995} = 3.6428$. Hence, from (B.4) for $p_{[2]} = 0.6$, $\Delta = 0.2$ we obtain

$$n = \frac{0.48}{(0.2)^2} \frac{(3.6428)^2}{2} = 6(3.6428)^2 = 79.62.$$

Also, for $p_{[2]} = 0.9$, $\Delta = 0.2$ we obtain

$$n = \frac{0.30}{(0.2)^2} \frac{(3.6428)^2}{2} = 3.75(3.6428)^2 = 49.76.$$

We note from Table 4.11B that the exact probabilities associated with $p_{[2]} = 0.6$, $\Delta = 0.2$, $n = 80$ and $p_{[2]} = 0.9$, $\Delta = 0.2$, $n = 50$ are 0.9946 and 0.9950, respectively; thus the agreement here with $P = 0.995$ is excellent.

Example B.2: Suppose that $k = 3$. Then (B.3) is a bivariate integral, and the associated correlation coefficient ρ which is given by (B.2) depends on the specified values of p and Δ . For $(p = 0.5, \Delta = 0.2)$ and $(p = 0.9, \Delta = 0.2)$ we have from (B.2) that $\rho = 0.543$ and $\rho = 0.600$, respectively. Then for $P = 0.95$ (say) we find from Table 1 of G-N-P [1973] that

$c_{3,P,\rho} = c_{3,0.95,0.543} = \sqrt{2} (1.908)$, and hence from (B.4) for $p_{[3]} = 0.5$, $\Delta = 0.2$ we obtain

$$n = \frac{0.46}{0.04} \frac{(\sqrt{2} (1.908))^2}{2} = 11.5(1.908)^2 = 41.9;$$

similarly, for $P = 0.95$ we find from Table 1 of G-N-P [1973] that

$c_{3,0.95,0.6} = \sqrt{2} (1.8997)$, and hence from (B.4) for $p_{[3]} = 0.9$, $\Delta = 0.2$ we obtain

$$n = \frac{0.30}{0.04} \frac{(\sqrt{2} (1.8997))^2}{2} = 7.5(1.8997)^2 = 27.1.$$

We note from Table 4.21B that the exact probabilities associated with $p_{[3]} = 0.5$, $\Delta = 0.2$, $n = 42$ and $p_{[3]} = 0.9$, $\Delta = 0.2$, $n = 28$ are 0.9469 and 0.9494, respectively; thus the agreement here with $P = 0.95$ is also excellent (even though the sample sizes are quite a bit smaller than the corresponding ones of Example B.1).

Appendix C: Recursion formulae for $E\{N\}$ for P^* when $k = 2$

At stage m denote $n - n_{i,m}$ and $z_{1,m} - z_{2,m}$ by m_i ($i = 1, 2$) and y , respectively, where $n_{i,m}$ and $z_{i,m}$ ($i = 1, 2; 1 \leq m \leq 2n-1$) are defined in the first paragraph of Section 2. Define

$$f = (n_{2,m} - z_{2,m}) - (n_{1,m} - z_{1,m}) = m_1 - m_2 + y.$$

It can be shown (see Lemma 4.1 of Kulkarni [1981]) that the expected total number of observations from stage m until termination depends only on (y, m_1, m_2) , i.e., only on (f, m_1, m_2) . It is clear that if Procedure P^* is used, the number of failures from the two populations differ at most by unity, i.e., $f = -1$ or 0 or 1 . Define

$$\begin{aligned} A(m_1, m_2) &= E\{N | (m_1, m_2), f = 1\} \\ &= \text{expected total number of observations required until} \\ &\quad \text{termination starting from } (m_1, m_2) \text{ when } f = 1. \end{aligned}$$

Similarly define

$$\begin{aligned} B(m_1, m_2) &= E\{N | (m_1, m_2), f = -1\} \\ \text{and} \\ C(m_1, m_2) &= E\{N | (m_1, m_2), f = 0\}. \end{aligned}$$

Then the following recursion formulae can be seen to hold for P^* :

For $m_1 \geq 2, m_2 \geq 0,$

$$A(m_1, m_2) = 1 + p_1 A(m_1 - 1, m_2) + (1 - p_1) C(m_1 - 1, m_2). \quad (C.1a)$$

For $m_1 \geq 0, m_2 \geq 2,$

$$B(m_1, m_2) = 1 + p_2 B(m_1, m_2 - 1) + (1 - p_2) C(m_1, m_2 - 1). \quad (C.1b)$$

For $m_1 \geq 1, m_2 \geq 1,$

$$C(m_1, m_2) = \begin{aligned} & 1 + p_1 C(m_1 - 1, m_2) + (1 - p_1) B(m_1 - 1, m_2) && \text{if } m_1 < m_2, \\ & 1 + p_2 C(m_1, m_2 - 1) + (1 - p_2) A(m_1, m_2 - 1) && \text{if } m_1 > m_2, \\ & 1 + (1/2)p_1 C(m_1 - 1, m_2) + (1/2)(1 - p_1) B(m_1 - 1, m_2) \\ & \quad + (1/2)p_2 C(m_1, m_2 - 1) + (1/2)(1 - p_2) A(m_1, m_2 - 1) && \text{if } m_1 = m_2. \end{aligned} \quad (C.1c)$$

The boundary conditions are

$$A(1, m_2) = 0, B(m_1, 1) = 0, C(0, m_2) = 0, C(m_1, 0) = 0. \quad (C.2)$$

Equations (C.1) and (C.2) can be solved recursively for any (m_1, m_2) to yield for any given n and $(p_{[1]}, p_{[2]})$

$$E\{N | p_{[1]}, p_{[2]}\} = C(n, n).$$

Thus by solving this one set of equations, $E\{N|(p_{[1]}, p_{[2]})\}$ can be found for all n . This method was used for computing the entries in Tables 4.7 and 4.11A.

In certain special cases the recursion equations can be solved to yield closed form expressions for $E\{N|(p_{[1]}, p_{[2]})\}$.

Case 1: $p_{[1]} = p$, $p_{[2]} = 1$. Then

$$E\{N|(p, 1)\} = \begin{cases} n + p(1-p^n)/2(1-p) - np^n/2 & \text{if } p < 1 \\ n & \text{if } p = 1. \end{cases}$$

Case 2: $p_{[1]} = 0$, $p_{[2]} = p$. Then

$$E\{N|(0, p)\} = \begin{cases} n(2-p) + p^n(1-p)/2(1-2p) \\ \quad - (1-p)^n(2-3p)/2(1-2p) & \text{if } p \neq 1/2 \\ 3n/2 - (n+1)/2^{n+1} & \text{if } p = 1/2. \end{cases}$$

In particular, $E\{N|(0, 0)\} = 2n-1$.

Remark C.1: Similar recursion formulae can be derived for

$E\{N|(p_{[1]}, p_{[2]}, \dots, p_{[k]})\}$ when $k > 2$. (See Section (5.6) in Kulkarni [1981].) Closed form expressions for $E\{N|(p_{[1]}, p_{[2]}, \dots, p_{[k]})\}$ can be obtained in certain special cases. For example when $p_{[1]} = p_{[t]} = 0$, $p_{[t+1]} = p_{[k]} = 1$ ($1 \leq t \leq k-1$) we have

Case 1: $1 \leq t \leq k-2$

$$E\{N\} = n + [(k-t)/k] \sum_{s=1}^t s \prod_{j=1}^s (t-j+1)/(k-j)$$

Case 2: $t = k-1$

$$E\{N\} = n + (k-2)(k+1)/2k.$$

Appendix D: An upper bound for $E\{N_{(i)}\}$ ($i < k$) for P^*

An upper bound for $E\{N_{(i)}\}$ ($i < k$) can be derived for $k \geq 2$ using the fact that for P^* , at any stage the numbers of failures from any two populations differ by at most one. Let $N_{(i)}^F$ denote the total number of failures at termination from the population associated with $p_{[i]}$ ($1 \leq i \leq k$). Then, in particular, we have

$$N_{(i)}^F \leq N_{(k)}^F + 1 \quad (i < k)$$

and hence

$$E\{N_{(i)}^F\} \leq E\{N_{(k)}^F\} + 1 \quad (i < k). \quad (D.1)$$

However,

$$E\{N_{(i)}^F\} = (1-p_{[i]})E\{N_{(i)}\} \quad (1 \leq i \leq k). \quad (D.2)$$

Combining (D.1) and (D.2), and noting that $E\{N_{(k)}\} \leq n$ we have for $i < k$ that

$$(1-p_{[i]})E\{N_{(i)}\} < 1 + (1-p_{[k]})E\{N_{(k)}\} \leq 1 + (1-p_{[k]})n,$$

and therefore for $p_{[i]} \neq 1$ ($i < k$) we have

$$E\{N_{(i)}\} \leq [1 + (1-p_{[k]})n]/(1-p_{[i]}). \quad (D.3)$$

Using (D.3) we have

$$E\{N\} = \sum_{i=1}^k E\{N_{(i)}\} \leq \sum_{i=1}^{k-1} [1 + (1-p_{[k]})n]/(1-p_{[i]}) + n. \quad (D.4)$$

The computational results in Tables 4.15 and 4.27 indicate that the bound (D.3) is quite good for large n as is seen from the following examples:

a) From Table 4.15 when $n = 100$ we have for $(p_{[1]} = 0.0, p_{[2]} = 0.2)$ that $E\{N_{(1)}\} = 80.000$ while (D.3) yields $E\{N_{(1)}\} \leq 81.0$; also from Table 4.15 when $n = 100$ we have for $(p_{[1]} = 0.7, p_{[2]} = 0.9)$ that $E\{N_{(1)}\} = 33.334$ while (D.3) yields $E\{N_{(1)}\} \leq 36.7$. For these same cases, from Table 4.11A we have $E\{N\} = 180.000$ and $E\{N\} = 133.332$ while (D.4) yields $E\{N\} \leq 181.0$ and $E\{N\} \leq 136.7$, respectively.

b) From Table 4.27 when $n = 40$ we have for $(p_{[1]} = 0.5, p_{[2]} = 0.6, p_{[3]} = 0.7)$ that $E\{N_{(1)}\} = 23.178$, $E\{N_{(2)}\} = 29.025$, while (D.3) yields $E\{N_{(1)}\} \leq 26.0$, $E\{N_{(2)}\} \leq 32.5$; also from Table 4.27 when $n = 40$ we have for $(p_{[1]} = 0.3, p_{[2]} = 0.6, p_{[3]} = 0.9)$ that $E\{N_{(1)}\} = 5.727$, $E\{N_{(2)}\} = 10.045$, while (D.3) yields $E\{N_{(1)}\} \leq 7.1$, $E\{N_{(2)}\} \leq 12.5$. For these same cases, from Table 4.24A we have $E\{N\} = 90.851$ and $E\{N\} = 55.761$ while (D.4) yields $E\{N\} \leq 98.5$ and $E\{N\} \leq 59.6$, respectively.

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→ In a recent paper, Bechhofer and Kulkarni proposed closed adaptive sequential procedures for a general class of k -population Bernoulli selection goals. These sequential selection procedures achieve the same probability of a correct selection, uniformly in the unknown single-trial "success" probabilities p_i ($1 \leq i \leq k$), as do the corresponding single-stage selection procedures which take exactly n observations from each of the k populations. The sequential procedures always require less (often substantially less) than kn observations to terminate experimentation. This earlier paper described the procedures, discussed their performance in general terms, and cited several of their optimality properties.

→ In the present paper we specialize these procedures, and focus on the particular goal of selecting the population associated with $p_{[k]}$ where $p_{[1]} \leq \dots \leq p_{[k]}$ are the ordered p_i ($1 \leq i \leq k$). We give exact numerical results for such performance characteristics of the sequential procedure (P^*) as the distribution of the total number of observations $N_{(i)}$ taken from the population associated with $p_{[i]}$ ($1 \leq i \leq k$), and the total number of observations $N = \sum_{i=1}^k N_{(i)}$ taken from all k populations, when the procedure terminates sampling. A simple upper bound for $E\{N_{(i)}\}$ ($i \neq k$) is given.

→ These results along with other related ones will assist the potential user of the sequential procedure in assessing its merits relative to those of other competing procedures.